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Foreword

This collection of technical reports addresses the following topics: threshold setting and required signal-to-noise ratios for combined normalization and or-ing; spectra and covariances for nonlinear processing in class A noise; exact performance of filtered and weighted energy detectors with mismatch; comparison of kernels for the modified Wigner function; solution of difference equation for wavenumber response of plate; detection performance of FSK in a partially-correlated fading medium.

Some of the material presented here is heavily based on the author's earlier work, which can be found in the following volumes in addition to the referenced technical reports:

*Performance of Detection and Communication Systems**,
*NUSC Scientific and Engineering Studies, 1974;

Spectral Estimation,
NUSC Scientific and Engineering Studies, 1977;

Coherence Estimation,
NUSC Scientific and Engineering Studies, 1979;

Receiver Performance Evaluation and Spectral Analysis,
NUSC Scientific and Engineering Studies, 1981;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1983;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1985;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1986;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1987;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1989;

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NUSC Scientific and Engineering Studies, 1990.

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Required Threshold Settings and Signal-to-Noise Ratios for Combined Normalization and Or-ing

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ABSTRACT

The system of interest here includes the combined nonlinear transformations of normalization and or-ing in the data processing chain. The required threshold settings for specified false alarm probabilities are determined, as well as the required input signal-to-noise ratios for specified detection probabilities. Specializations to normalization or or-ing alone are available by particular choices of parameter values built into the models.

When the sizes of the transformations are changed, that is, the amount of normalization and/or or-ing, the thresholds will also have to be changed if the earlier probabilities are to be maintained; these options are included in the first program listed. In addition, the possibility of changing the number of thresholds, at the same time that the sizes of the nonlinear transformations are changed, is included in the second program listed. Furthermore, four alternatives are allowed for selecting the spacing of the thresholds, namely, equispaced in power, equispaced in decibels, preset normalized thresholds, or preset probabilities.

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LIST OF SYMBOLS

RV	random variable
IID	independent identically distributed
PDF	probability density function
CDF	cumulative distribution function
CF	characteristic function
Prob	probability
x_m	m-th noise-only random variable, (1)
M	M+1 is the number of random variables $\{x_m\}$
$p_x(u)$	probability density function of x_m, (1)
$C_x(X)$	cumulative distribution function of x_m, (1)
$f_x(\xi)$	characteristic function of x_m, (2)
$\mu_x(j)$	j-th moment of x_m
$\chi_x(j)$	j-th cumulant of x_m
$\tilde{C}_x(P)$	inverse cumulative distribution function for x, (3)
f	nonlinear transformation, (5)
y	output of nonlinear transformation, (5)
w_{km}	inputs to normalizer and or-ing device, (8)
x_k	output from or-ing device, (8)
K	size of or-ing operation, (8)
$C_y(Y)$	cumulative distribution function of y, (9)
Y_1	first threshold for application to y, (9)
P_1	specified cumulative probability at Y_1, (9)
$\tilde{C}_y(P)$	inverse cumulative distribution function for y, (9)
Y_T	last threshold for application to y, (10)
T	total number of thresholds, (10)

A_y	scale factor, (10)
dB_y	decibel ratio between Y_T and Y_1, (10)
Y_t	t-th threshold for application to y, (11)
$\underline{Y}_t$	t-th normalized threshold for y, (12)
μ_y	mean of random variable y, (12)
σ_y	standard deviation of random variable y, (12)
P_t	cumulative probability value at threshold Y_t, (13)
$1-P_t$	false alarm probability, (13)
D_t	t-th threshold in decibels, (14), (16)
N	new size of nonlinear transformation, (20)
z	output of new nonlinear transformation, (20)
$C_z(Z)$	cumulative distribution function of z, (21)
Z_t	t-th threshold for application to z, (21)
$\tilde{C}_z(P)$	inverse cumulative distribution function for z, (21)
$\underline{Z}_t$	t-th normalized threshold for z, (22)
μ_z	mean of random variable z, (22)
σ_z	standard deviation of random variable z, (22)
dB_z	decibel ratio between Z_T and Z_1, (23)
U	new number of thresholds, (24)
Z'_u	u-th new threshold, (24)
P'_u	cumulative probability at Z'_u, (25)
$\underline{Z}'_u$	u-th normalized threshold for z, (26)
$C_y(Y;R)$	cumulative distribution of y with signal present, (27)
R	input power signal-to-noise ratio, (27)
S_1	specified probability, (27)
R_t	required signal-to-noise ratio at threshold Y_t, (27)

Pd_j	j-th detection probability, (30)
a	average of M random variables, (31)
$f_a(\xi)$	characteristic function of a, (32), (36)
$p_a(u)$	probability density function of a, (32), (37)
$p_y(u)$	probability density function of y, (34), (39)
$\overline{y^j}$	j-th moment of random variable y, (41)
$(b)_j$	$b(b+1)\cdots(b+j-1)$, (41)
$\chi_y(j)$	j-th cumulant of y, (55)
a_k	k-th denominator average, (62)
F_1, F_2	auxiliary constants, (79)
b_k	k-th denominator average, (81)
L	new size of or-ing device, (82)
G_1, G_2	auxiliary constants, (86)
Q_t	auxiliary function, (95), (100)

REQUIRED THRESHOLD SETTINGS AND SIGNAL-TO-NOISE RATIOS FOR COMBINED NORMALIZATION AND OR-ING

INTRODUCTION

When weak signals of unknown strength and location have to be detected in the presence of noise of unknown and varying level, it is necessary to make estimates of the intensity of the interfering background. These estimated (noise) levels are then compared with that for a candidate signal level and location for purposes of making statements about the likelihood of signal presence or absence in that particular data segment under investigation. Here, we will investigate the performance capability of such a normalizer, in terms of the false alarm and detection probabilities, and determine the thresholds and input signal-to-noise ratios required to attain these probabilities.

Furthermore, when large amounts of multichannel data have to be processed with limited computational facilities in reasonable or short intervals of time, it is necessary to resort to shortcuts or data reduction in order to avoid overload. One possible approach is to employ or-ing, in which only the largest of a set of quantities is retained for further data processing and decision making.

Finally, in practice, it is often necessary to utilize both normalization and or-ing together. This combination of nonlinear processors requires a resetting of the thresholds that would have been appropriate for use of the normalizer alone. Here, we shall

investigate all three situations, namely normalization, or-ing, and a combination of normalization and or-ing, in terms of the probabilities and thresholds stated above.

When the size of the nonlinear transformation, whether it be normalization, or-ing, or both, is changed, the required thresholds will have to be changed if the previously realized (false alarm) probabilities are to be maintained. For example, suppose we had been or-ing 8 random variables and decided to change the or-ing size to accept 16 random variables instead, for purposes of further data reduction. Then, the required threshold settings would have to be modified to maintain specified false alarm probabilities, as would the required input signal-to-noise ratios for specified detection probabilities. This maintenance of probabilities under a change of size of transformation will be investigated here.

At the same time that the size of a nonlinear transformation is changed, it may be desired to subject its output to a different number of thresholds than were utilized previously. This possibility is allowed and analyzed in addition.

The particular physical situation considered here is that of multiple simultaneous beamformer outputs, each with large banks of narrowband filters subject to envelope-squared detection. For Gaussian noise inputs, the probability density function of each of these device outputs is exponential. Furthermore, when a Gaussian signal is also present at the input of one of these narrowband filters, the corresponding probability density

function of the detected output is still exponential, but with a level governed by the signal-to-noise ratio at that particular filter output. This scenario will be the mainstay of the analysis here.

The physical motivation for this study is to be able to set requantization levels on a display, in order to achieve constant marking density independent of the signal processing parameters such as the amount of or-ing and normalizer size. Such displays occur in active as well as passive sonar systems.

DEFINITIONS OF FUNCTIONS

The random variables (RV) for noise-only, $\{x_m\}$ for $0 \le m \le M$, are independent and identically distributed (IID) with common probability density function (PDF) $p_x(u)$ and cumulative distribution function (CDF) $C_x(X)$, where

$$C_x(X) = \text{Prob}(x_m < X) = \int_{-\infty}^{X} du\; p_x(u) \; . \qquad (1)$$

The corresponding characteristic function (CF) of RV x_m is

$$f_x(\xi) = \int du\; \exp(i\xi u)\; p_x(u) \; , \qquad (2)$$

where integrals without limits are over $-\infty,+\infty$. The moments and cumulants of order j of RV x_m are denoted by $\mu_x(j)$ and $\chi_x(j)$, respectively. The inverse function to CDF $C_x(X)$ in (1) is denoted as $\tilde{C}_x(P)$; that is,

$$\text{if } P = C_x(X), \text{ then } X = \tilde{C}_x(P) \text{ for } 0 < P < 1 \; . \qquad (3)$$

As an example, consider narrowband filter outputs for which

$$p_x(u) = \exp(-u) \quad \text{for } u > 0 \; ,$$

$$C_x(X) = 1 - \exp(-X) \quad \text{for } X > 0 \; ,$$

$$f_x(\xi) = (1 - i\xi)^{-1} \quad \text{for all } \xi \; ,$$

$$\tilde{C}_x(P) = -\ln(1 - P) \quad \text{for } 0 < P < 1 \; ,$$

$$\mu_x(j) = j! \; , \quad \chi_x(j) = (j-1)! \quad \text{for } j \ge 1 \; . \qquad (4)$$

The scaling of RV x_m in (4) has been taken such that its mean is 1. This is done solely for notational convenience; it will not affect the probabilities realized herein nor the required signal-to-noise ratios. However, it does influence the threshold settings calculated, which would have to be scaled for a different input noise level.

THRESHOLD RESETTING

A collection of IID RVs, $\{x_m\}$ for $0 \le m \le M$, is subject to a nonlinear transformation f, yielding output

$$y = f(x_0, x_1, \ldots, x_M) \ . \tag{5}$$

For example, a normalizer is characterized by

$$y = \frac{x_0}{\frac{1}{M}\left(x_1 + x_2 + \cdots + x_M\right)} \ , \quad M \ge 1 \ , \tag{6}$$

while an or-ing device yields

$$y = \max(x_1, x_2, \ldots, x_M) \ , \quad M \ge 1 \ . \tag{7}$$

A combination of a normalizer and or-ing device will require a more general formulation; then we would use

$$x_k = \frac{w_{ko}}{\frac{1}{M}\sum_{m=1}^{M} w_{km}} \quad \text{for } 1 \le k \le K \ ,$$

$$y = \max(x_1, x_2, \ldots, x_K) \ , \tag{8}$$

where we need two parameters, M and K, and $\{w_{km}\}$ are K(M+1) IID RVs. Of course, then the K RVs $\{x_k\}$ are also IID.

Let $C_y(Y)$ be the CDF of RV y at the output of general nonlinearity f in (5). Choose threshold Y_1 such that specified cumulative probability P_1 is realized there; that is,

$$P_1 = C_y(Y_1) \;, \quad \text{or} \quad Y_1 = \tilde{C}_y(P_1) \;, \tag{9}$$

where $\tilde{C}_y$ is the inverse function to C_y. $1-P_1$ is the false alarm probability at threshold Y_1. Also, choose additional thresholds $\{Y_t\}$ such that $Y_1 < Y_2 < \cdots < Y_T$, for a total of T thresholds, with the largest one being

$$Y_T = Y_1 \, A_y \;, \quad \text{where} \quad dB_y = 10 \log_{10}(A_y) \tag{10}$$

is a specified decibel value. Then take the remaining thresholds according to equal spacing rule

$$Y_t = Y_1 + \frac{Y_T - Y_1}{T - 1}(t - 1) \quad \text{for } 1 \le t \le T \;. \tag{11}$$

The normalized thresholds for RV y are defined as

$$\underline{Y}_t = \frac{Y_t - \mu_y}{\sigma_y} \quad \text{for } 1 \le t \le T \;. \tag{12}$$

where μ_y and σ_y are the mean and standard deviation of RV y.

Then compute the cumulative probabilities realized at these thresholds $\{Y_t\}$, namely

$$P_t = C_y(Y_t) \quad \text{for } 1 \le t \le T \;, \tag{13}$$

and print out M, dB_y, μ_y, σ_y, T, $\{Y_t\}$, $\{\underline{Y}_t\}$, $\{P_t\}$. The false alarm probabilities are $\{1-P_t\}$.

An alternative choice is to space the thresholds $\{Y_t\}$ equally in decibels. That is, defining

$$D_t = 10 \log_{10}(Y_t) \quad \text{for } 1 \le t \le T \tag{14}$$

as the threshold in decibels, we take D_1 as given in terms of Y_1 and we take

$$D_T = D_1 + dB_y \ . \tag{15}$$

The intermediate decibel thresholds are then selected according to the equal spacing rule

$$D_t = D_1 + \frac{D_T - D_1}{T - 1}(t - 1) \quad \text{for } 1 \le t \le T \ . \tag{16}$$

The power thresholds can then be evaluated as

$$Y_t = 10^{D_t/10} \quad \text{for } 1 \le t \le T \ . \tag{17}$$

The cumulative probabilities at these latter power thresholds follow from (13) as before.

Another possibility is to simply set the thresholds according to

$$Y_t = \mu_y + \sigma_y \, \underline{Y}_t \quad \text{for } 1 \le t \le T \ , \tag{18}$$

where normalized thresholds $\{\underline{Y}_t\}$ are preset constants determined by the user. In this latter case, probability P_1 in (9) would not be realized at initial threshold $Y_1 = \mu_y + \sigma_y \, \underline{Y}_1$. In any event, the desired printouts are the quantities listed under (13).

Finally, we could set the thresholds such that preset values of probabilities $\{P_t\}$ are realized for all $1 \le t \le T$. That is, solve (13) for the thresholds according to

$$Y_t = \tilde{C}_y(P_t) \quad \text{for } 1 \le t \le T \; . \tag{19}$$

This amounts to setting T different false alarm probabilities $\{1-P_t\}$.

ALTERNATIVE SIZE TRANSFORMATION

Now consider the new RV z obtained by changing the parameter value from M to N in the given nonlinear transformation in (5):

$$z = f(x_0, x_1, \ldots, x_N) \quad . \tag{20}$$

N can be larger or smaller than M. Let the CDF of RV z be $C_z(Z)$. We now choose new thresholds, $\{Z_t\}$ for $1 \le t \le T$, such that the probabilities $\{P_t\}$ in (13) are maintained for RV z; that is, we choose thresholds $\{Z_t\}$ for RV z in (20) such that

$$C_z(Z_t) = P_t \; , \quad \text{or } Z_t = \tilde{C}_z(P_t) \quad \text{for } 1 \le t \le T \; . \tag{21}$$

These thresholds $\{Z_t\}$, to be employed for RV z, will not necessarily have equal spacing, as did the thresholds $\{Y_t\}$ in (11), for example, for RV y. The normalized thresholds for RV z are

$$\underline{Z}_t = \frac{Z_t - \mu_z}{\sigma_z} \quad \text{for } 1 \le t \le T \; , \tag{22}$$

where μ_z and σ_z are the mean and standard deviation of RV z. Then print out N, dB_z, μ_z, σ_z, T, $\{Z_t\}$, $\{\underline{Z}_t\}$, $\{P_t\}$, where

$$dB_z \equiv 10 \log_{10}(Z_T/Z_1) \quad . \tag{23}$$

DIFFERENT NUMBER OF THRESHOLDS

If RV z is to be subject to a different number, U, of thresholds than RV y was, it is not always reasonable to try to maintain the set of T probabilities $\{P_t\}$ realized in (13). (U can be larger or smaller than T.) One alternative is to maintain the edge probabilities P_1 and P_T according to (21), thereby determining values for Z_1 and Z_T. Then choose a different complete set of thresholds $\{Z'_u\}$ for RV z according to equal spacing rule

$$Z'_u = Z_1 + \frac{Z_T - Z_1}{U - 1}(u - 1) \quad \text{for } 1 \le u \le U \ . \tag{24}$$

We can then evaluate the cumulative probabilities at these latter threshold values as

$$P'_u = C_z(Z'_u) \quad \text{for } 1 \le u \le U \ . \tag{25}$$

Of course, $P'_1 = P_1$ and $P'_U = P_T$, since $Z'_1 = Z_1$ and $Z'_U = Z_T$. This also means that dB_z is still given by (23).

It must be noted that this change in philosophy, namely to maintain only edge probabilities P_1 and P_T, will not reduce to the earlier results when we set U = T here. The new thresholds in z, given by (24), are equally spaced even if U = T, whereas the former thresholds given by (21) are not.

Print out N, dB_z, μ_z, σ_z, U, $\{Z'_u\}$, $\{\underline{Z}'_u\}$, $\{P'_u\}$, where $\{\underline{Z}'_u\}$ are the normalized thresholds

$$\underline{Z}'_u = \frac{Z'_u - \mu_z}{\sigma_z} \quad \text{for } 1 \le u \le U \ . \tag{26}$$

SIGNAL-TO-NOISE RATIO REQUIREMENTS

Let us return to the original nonlinear transformation (5) characterized by parameter M. The CDF of RV y, with signal present in just <u>one</u> of the input RVs $\{x_m\}$, is denoted by $C_y(Y;R)$, where R is the input signal-to-noise ratio (SNR) in that particular x_m RV containing a signal. If we want this new CDF to realize a specified probability value S_1 at all the thresholds $\{Y_t\}$ in (11), the required SNRs $\{R_t\}$ must satisfy

$$S_1 = C_y(Y_t;R_t) \quad \text{for } 1 \le t \le T \ . \tag{27}$$

From (13) and (27), we know that

$$P_t = C_y(Y_t) = C_y(Y_t;0) \quad \text{for } 1 \le t \le T \ . \tag{28}$$

Therefore, specified probability S_1 in (27) must satisfy

$$S_1 \le P_t \quad \text{for } 1 \le t \le T \ , \tag{29}$$

in order that nonnegative SNRs $\{R_t\}$ can result as solutions to (27). That is, each cumulative distribution value must be decreased from P_t to S_1, meaning that each exceedance probability has been increased from false alarm probability value $1-P_t$ to detection probability value $Pd_1 = 1-S_1$. The actual determination of CDF $C_y(Y;R)$ will have to wait until after we have specified and analyzed the nonlinear transformations (5) of interest, for the case of noise-only present.

If several detection probabilities, like $Pd_1 = .5$, $Pd_2 = .9$, $Pd_3 = .99$, are specified, there will be a set of equations like (27) for each case, namely

$$1 - Pd_j = S_j = C_y(Y_t;R_t) \quad \text{for } 1 \le t \le T \; ; \quad j = 1,2,3 \; . \qquad (30)$$

STATISTICS OF NORMALIZER

Let RV y be obtained by means of a normalization procedure from IID RVs $\{x_m\}$, $0 \le m \le M$, according to transformation

$$y = \frac{x_0}{\frac{1}{M}\left(x_1 + \cdot \cdot \cdot + x_M\right)} \equiv \frac{x_0}{a} \ , \tag{31}$$

where we assume that $x_m \ge 0$ for all m. The average RV, a, in the denominator of (31) has, respectively, CF and PDF

$$f_a(\xi) = [f_x(\xi/M)]^M \ ,$$

$$p_a(u) = \frac{1}{2\pi} \int d\xi \ \exp(-iu\xi) \ [f_x(\xi/M)]^M \ . \tag{32}$$

The CDF of RV y in (31) is, using the statistical independence of x_0 and a,

$$C_y(Y) = \text{Prob}(y < Y) = \text{Prob}(x_0 < Ya) =$$

$$= \int_0^\infty dt \ p_a(t) \int_0^{Yt} du \ p_x(u) = \int_0^\infty dt \ p_a(t) \ C_x(Yt) \ . \tag{33}$$

The corresponding PDF of RV y is, upon differentiation,

$$p_y(u) = \int_0^\infty dt \ t \ p_a(t) \ p_x(ut) \ . \tag{34}$$

The moments of RV y can be found from a couple of forms:

$$\mu_y(j) = \overline{y^j} = \int_0^\infty du\ u^j\ p_y(u) =$$

$$= \overline{x_0^j}\ \overline{a^{-j}} = \mu_x(j)\ \mu_a(-j) = \int du\ u^j\ p_x(u) \times \int dt\ t^{-j}\ p_a(t)\ . \quad (35)$$

Convergence of the last integral in (35) may not occur for larger values of j; that is, due to the division in (31), RV y may not have finite higher-order moments.

EXAMPLE

The example presented in (4) is utilized here. From (4) and (32), the CF and PDF of average RV a in (31) are

$$f_a(\xi) = \frac{1}{(1 - i\xi/M)^M} \quad (36)$$

and

$$p_a(u) = \frac{M^M\ u^{M-1}\ \exp(-Mu)}{(M-1)!} \quad \text{for } u > 0\ , \quad (37)$$

respectively. The CDF of RV y then follows from (33) and (4) as

$$C_y(Y) = \int_0^\infty dt\ \frac{M^M\ t^{M-1}\ \exp(-Mt)}{(M-1)!}\ [1 - \exp(-Yt)] =$$

$$= 1 - \left(1 + \frac{Y}{M}\right)^{-M} \quad \text{for } Y > 0\ . \quad (38)$$

The PDF of RV y is therefore

$$p_y(u) = \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0 \ . \tag{39}$$

The inverse function to CDF $C_y(Y)$ in (38) is

$$\tilde{C}_y(P) = M\left[(1 - P)^{-1/M} - 1\right] \quad \text{for } 0 < P < 1 \ . \tag{40}$$

The j-th moment of RV y is given by

$$\mu_y(j) = \overline{y^j} = \int du\ u^j\ p_y(u) = \int_0^\infty du\ u^j \left(1 + \frac{u}{M}\right)^{-M-1} =$$

$$= M^{j+1}\ B(j+1,M-j) = M^{j+1} \frac{\Gamma(j+1)\ \Gamma(M-j)}{\Gamma(M+1)} = \frac{j!\ M^j}{(M-j)_j} \quad \text{for } j < M \ , \tag{41}$$

where we used (39) and [1; 3.194 3 and 8.384 1]. In particular,

$$\overline{y} = \frac{M}{M-1}\ , \quad \overline{y^2} = \frac{2M^2}{(M-2)(M-1)}\ , \quad \sigma_y^2 = \frac{M^3}{(M-2)(M-1)^2} \quad \text{for } M > 2 \ . \tag{42}$$

The condition $M > 2$ is necessary for RV y to possess a finite variance. We now have all the quantities required for application in the previous section on threshold resetting.

For $M = \infty$, $f_a(\xi)$ in (36) equals $\exp(i\xi)$; that is, RV a in (31) is equal to the constant 1. This corresponds to no normalization and, in fact, (31) reduces to $y = x_o$. Also, (38) - (41) reduces to (4), as expected.

CHANGE IN SIZE OF NORMALIZER

If we now change from size M to N in normalizer (31), we obtain RV z as considered in (20) and sequel. Its CDF follows, by similarity of form to (38), as

$$C_z(Z) = 1 - \left(1 + \frac{Z}{N}\right)^{-N} \quad \text{for } Z > 0 \ . \qquad (43)$$

Its inverse function is

$$\tilde{C}_z(P) = N\left[(1 - P)^{-1/N} - 1\right] \quad \text{for } 0 < P < 1 \ . \qquad (44)$$

Reference to (42) also reveals that the mean and standard deviation of RV z are

$$\mu_z = \frac{N}{N-1} \ , \qquad \sigma_z = \frac{N}{N-1}\left(\frac{N}{N-2}\right)^{\frac{1}{2}} \quad \text{for } N > 2 \ . \qquad (45)$$

The new thresholds are given by (21) and (22).

STATISTICS OF OR-ING DEVICE

Let $\{x_m\}$, $1 \le m \le M$, be IID RVs with common PDF $p_x(u)$ and CDF $C_x(X)$. The CDF and PDF of the maximum RV

$$y = \max\{x_1, x_2, \ldots, x_M\} , \tag{46}$$

yielded by an or-ing device, are then

$$C_y(Y) = [C_x(Y)]^M , \tag{47}$$

$$p_y(u) = M \, [C_x(u)]^{M-1} \, p_x(u) , \tag{48}$$

respectively. The inverse function to CDF C_y in (47) is

$$\tilde{C}_y(P) = \tilde{C}_x\left(P^{1/M}\right) \quad \text{for } 0 < P < 1 . \tag{49}$$

Here, again, $\tilde{C}_x$ is the inverse function to CDF C_x of RV x_m. The CF of RV y follows from (48) as

$$f_y(\xi) = \int du \, \exp(i\xi u) \, M \, [C_x(u)]^{M-1} \, p_x(u) . \tag{50}$$

The moments of RV y may be obtained from (48) in the form

$$\mu_y(j) = \int du \, u^j \, M \, [C_x(u)]^{M-1} \, p_x(u) . \tag{51}$$

Alternatively, the cumulants can be obtained by a power series expansion of the natural logarithm of CF $f_y(\xi)$ in (50).

EXAMPLE

We again use the results given in (4). Substitution into (50) yields

$$f_y(\xi) = \int_0^\infty du \exp(i\xi u - u)\; M\; [1 - \exp(-u)]^{M-1} =$$

$$= M \int_0^1 dt\; t^{-i\xi}\; (1-t)^{M-1} = \frac{\Gamma(1-i\xi)\;\Gamma(M+1)}{\Gamma(M+1-i\xi)} =$$

$$= \left[\prod_{m=1}^{M}\left(1 - \frac{i\xi}{m}\right)\right]^{-1} , \tag{52}$$

where we let $t = \exp(-u)$ and used [1; 8.380 1 and 8.384 1]. This is a very compact form and is easily computed numerically, if necessary. This result illustrates that RV y has the same statistics as RV w defined by

$$w \equiv \sum_{m=0}^{M} \frac{x_m}{m} = x_1 + \frac{1}{2}x_2 + \cdots + \frac{1}{M}x_M . \tag{53}$$

In order to determine the cumulants of RV y, we consider

$$\ln f_y(\xi) = - \sum_{m=1}^{M} \ln\left(1 - \frac{i\xi}{m}\right) = \sum_{m=1}^{M} \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{i\xi}{m}\right)^k . \tag{54}$$

The cumulants follow immediately as

$$\chi_y(j) = (j-1)! \sum_{m=1}^{M} \frac{1}{m^j} \quad \text{for } j \ge 1 \ . \tag{55}$$

In particular, the mean and variance of RV y are

$$\mu_y = \chi_y(1) = \sum_{m=1}^{M} \frac{1}{m} \ , \quad \sigma_y^2 = \chi_y(2) = \sum_{m=1}^{M} \frac{1}{m^2} \ , \tag{56}$$

respectively. It is seen that the mean of RV y increases without limit, in fact, logarithmic with M. On the other hand, the variance saturates at $\pi^2/6$, meaning that the standard deviation σ_y of RV y cannot exceed $\pi/6^{\frac{1}{2}} = 1.283$.

The CDF of y follows from (47) and (4) as

$$C_y(Y) = [1 - \exp(-Y)]^M \quad \text{for } Y > 0 \ , \tag{57}$$

with corresponding inverse function

$$\tilde{C}_y(P) = -\ln\left(1 - P^{1/M}\right) \quad \text{for } 0 < P < 1 \ . \tag{58}$$

For M = 1, there is no or-ing, and (46) reduces to $y = x_1$. Also, (52), (55), (57), and (58) reduce to (4), as expected.

CHANGE IN SIZE OF OR-ING DEVICE

If we change from size M to N in or-ing device (46), we obtain RV z considered in (20) and sequel. Its CDF is, by similarity in form to (57),

$$C_z(Z) = [1 - \exp(-Z)]^N \quad \text{for } Z > 0 . \tag{59}$$

The corresponding inverse function is

$$\tilde{C}_z(P) = - \ln\left(1 - P^{1/N}\right) \quad \text{for } 0 < P < 1 . \tag{60}$$

Reference to (56) reveals that the mean and variance of RV z are given by

$$\mu_z = \sum_{n=1}^{N} \frac{1}{n} , \qquad \sigma_z^2 = \sum_{n=1}^{N} \frac{1}{n^2} . \tag{61}$$

The new thresholds for RV z are given by (21) and (22).

STATISTICS OF NORMALIZER AND OR-ING DEVICE

Here, we consider a set of K(M+1) IID RVs $\{w_{km}\}$ subject to both normalization and or-ing, according to

$$a_k = \frac{1}{M}\sum_{m=1}^{M} w_{km} , \quad x_k = \frac{w_{ko}}{a_k} \quad \text{for } 1 \le k \le K , \tag{62}$$

$$y = \max(x_1, x_2, \ldots, x_K) . \tag{63}$$

The statistics of the normalization portion, namely $\{x_k\}$ for $1 \le k \le K$, were previously considered in (31) - (35) for the general case, and then specialized to an example in (36) - (42). Also, the analysis of the or-ing portion was conducted in (46) - (51) and then specialized to an example in (52) - (58). We will rely heavily on those results in order to minimize the presentation in this section.

EXAMPLE

We presume that all the input RVs $\{w_{km}\}$ in (62) have the common exponential PDF used in example (4) for all the earlier results here. Then, by reference to (31) and (39), we can state that the common PDF of RVs $\{x_k\}$ defined in (62) is

$$p_x(u) = \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0 . \tag{64}$$

Similarly, the CDF of RVs $\{x_k\}$ follows, by reference to (38), as

$$C_x(X) = 1 - \left(1 + \frac{X}{M}\right)^{-M} \quad \text{for } X > 0 . \tag{65}$$

The CDF of RV y defined in (63) is then

$$C_y(Y) = [C_x(Y)]^K = \left[1 - \left(1 + \frac{Y}{M}\right)^{-M}\right]^K \quad \text{for } Y > 0 . \tag{66}$$

Its inverse function follows readily as

$$\tilde{C}_y(P) = M\left[\left(1 - P^{1/K}\right)^{-1/M} - 1\right] \quad \text{for } 0 < P < 1 . \tag{67}$$

The PDF of RV y can be obtained by differentiation of (66):

$$p_y(u) = K\ [C_x(u)]^{K-1}\ p_x(u) =$$

$$= K \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1} \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0 . \tag{68}$$

The j-th moment of RV y can be found from the integral

$$\mu_y(j) = \int_0^\infty du\ u^j\ p_y(u) \quad \text{for } j < M . \tag{69}$$

where we have taken advantage of the fact that RV y can never be negative. However, a useful alternative in some cases is afforded by employing integration by parts on (69), with the result, for $j \geq 1$, that

$$\mu_y(j) = j \int_0^\infty du\, u^{j-1} \, [1 - C_y(u)] \; , \tag{70}$$

where we assume that

$$\lim_{u \to +\infty} u^j [1 - C_y(u)] = 0 \; . \tag{71}$$

This requirement is tantamount to presuming that the j-th moment $\mu_y(j)$ exists, that is, $j < M$.

When we use CDF (66) for RV y, then j-th moment (70) becomes

$$\mu_y(j) = j \int_0^\infty du\, u^{j-1} \left\{1 - \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^K\right\} . \tag{72}$$

In order to evaluate this integral, let, for the moment,

$$Q = \left(1 + \frac{u}{M}\right)^{-M} . \tag{73}$$

Then, the bracketed term in (72) can be expanded according to

$$[1 - Q]^K = \sum_{k=0}^{K} \binom{K}{k} (-Q)^k = 1 + \sum_{k=1}^{K} (-1)^k \binom{K}{k} \left(1 + \frac{u}{M}\right)^{-Mk} . \tag{74}$$

Employment of this result in (72) yields

$$\mu_y(j) = j \int_0^\infty du\ u^{j-1} \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \left(1 + \frac{u}{M}\right)^{-Mk} =$$

$$= j \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \int_0^\infty du\ u^{j-1} \left(1 + \frac{u}{M}\right)^{-Mk} =$$

$$= j!\ M^j \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k-1}}{(Mk-j)_j} \quad \text{for } 1 \le j < M , \tag{75}$$

where we employed [1; 3.194 3 and 8.384 1] and simplified the end result. For $K = 1$, (75) reduces to (41), as it must.

The first two moments for RV y follow readily from (75) as

$$\mu_y(1) = M \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k-1}}{Mk-1} \tag{76}$$

and

$$\mu_y(2) = 2M^2 \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k-1}}{(Mk-2)(Mk-1)} \quad \text{for } M > 2 . \tag{77}$$

Both of these results are easily computed by means of recurrences; however, a bad feature of these two sums is that they are alternating series and lose significance for large K due to the binomial coefficient which gets very large. A backup procedure is to revert to numerical integration of (68) - (69) or (70) - (72), both of which integrands can never go negative and which decay as u^{j-1-M} for large u.

However, better alternatives to the first and second moments have been found, that are not subject to cancellation and loss of significance. Namely, it is shown in appendix A that

$$\mu_y(1) = M[F_1 - 1] \ , \qquad \sigma_y = M\left(F_2 - F_1^2\right)^{\frac{1}{2}} \ , \tag{78}$$

where

$$F_1 = \prod_{k=1}^{K} \left\{ \frac{k}{k - \frac{1}{M}} \right\} \ , \qquad F_2 = \prod_{k=1}^{K} \left\{ \frac{k}{k - \frac{2}{M}} \right\} \quad \text{for } M > 2 \ . \tag{79}$$

These finite products are very useful and retain significance even for large K, where (76) and (77) are useless. The very compact BASIC program listed below computes both quantities in (78). The program has been written so as to avoid overflow, even when K is large.

```
K=
M=            !  M > 2
A=1/M
B=2/M
F1=F2=1
FOR Ks=1 TO K
F1=F1*Ks/(Ks-A)
F2=F2*Ks/(Ks-B)
NEXT Ks
Muy=M*(F1-1)
Sigy=M*SQR(F2-F1*F1)
```
(80)

By using the techniques in appendix A and laboriously expanding out (75) for j = 3, it has been found that

$$\mu_y(3) = M^3(F_3 - 3F_2 + 3F_1 - 1) \ ,$$

where we define products

$$F_m = \prod_{k=1}^{K} \left\{ \frac{k}{k - \frac{m}{M}} \right\} \quad \text{for } 0 \le m < M \; .$$

Notice that $F_0 = 1$. Based on this result and (78), we conjectured that the j-th moment is generally given by

$$\mu_y(j) = M^j \sum_{n=0}^{j} (-1)^n \binom{j}{n} F_{j-n} \quad \text{for } j < M \; .$$

In fact, this has been confirmed numerically for several values of M, K, and j. For large j, the alternating character of this series would also suffer from loss of significance; however, for the low order moments of general interest, this is not a problem.

The third and fourth cumulants of RV y were also computed in terms of sequence $\{F_n\}$; they turned out to be, for $M > 4$,

$$\chi_y(3) = M^3 \left[F_3 - 3F_2F_1 + 2F_1^3 \right] \; ,$$

$$\chi_y(4) = M^4 \left[F_4 - 4F_3F_1 - 3F_2^2 + 12F_2F_1^2 - 6F_1^4 \right] \; .$$

But these rules for obtaining cumulants $\{\chi_y(j)\}$ from products $\{F_n\}$ are <u>identical</u> to the general rules for obtaining cumulants from moments, within the factor M^j; see, for example, [4; page 70, (3.41)]. Thus, we have a very efficient and accurate method for obtaining the low-order cumulants directly from the finite products $\{F_n\}$. The case for $\chi_y(1)$ in (78) differs slightly from the usual rule, in the need to subtract 1. (See appendix C for analytic corroboration of these conjectures.)

ANALYTICAL CHECKS

Numerous checks on the results above are possible. When $M = \infty$ (no normalization), the CDF of y in (66) reduces to (57), where it must be remembered that K in this section on combined normalization and or-ing corresponds to M in the section on or-ing alone. On the other hand, if $K = 1$ (no or-ing) in (66), the result in (38) correctly emerges.

With regard to the inverse CDF in (67), it reduces to (58) for $M = \infty$, whereas it reduces to (40) for $K = 1$. The PDF of RV y, given in (68), reduces to the derivative of (57) for $M = \infty$, whereas it reduces to (39) for $K = 1$. Finally, the first two moments in (78) - (79) reduce, after some manipulations, to (56) for $M = \infty$; on the other hand, the j-th moment for $K = 1$ is best handled from form (75) which correctly reduces to (41).

EXTENSIONS

The case where the normalizer and or-ing device are followed by an averager is discussed in the summary below, and the method of determining the performance of that system is outlined. Another alternative with practical merit is that of normalization followed by averaging and or-ing. Since the CF of the normalizer output is available by a Fourier transform of (39), it can be raised to a power to find the CF at the averager output. Then another Fourier transform can yield the CDF. At this point, we could utilize (47) to find the CDF of the system output.

CHANGES IN SIZES OF NORMALIZER AND OR-ING DEVICE

We now address the change in size of the normalizer in (62) from size M to N and the change in size of the or-ing device in (63) from K to L. There are no restrictions on the relative sizes of the parameters. The new equations are

$$b_k = \frac{1}{N}\sum_{n=1}^{N} w_{kn}\ , \qquad y_k = \frac{w_{ko}}{b_k} \quad \text{for } 1 \le k \le L\ , \tag{81}$$

$$z = \max(y_1, y_2, \ldots, y_L)\ . \tag{82}$$

The CDF of RV z follows, by similarity of form to (66), as

$$C_z(Z) = \left[1 - \left(1 + \frac{Z}{N}\right)^{-N}\right]^L \quad \text{for } Z > 0\ . \tag{83}$$

The inverse function is easily shown to be

$$\tilde{C}_z(P) = N\left[\left(1 - P^{1/L}\right)^{-1/N} - 1\right] \quad \text{for } 0 < P < 1\ . \tag{84}$$

The first two moments of RV z in (82) are given, by comparison with (78) and (79), as

$$\mu_z(1) = N[G_1 - 1]\ , \qquad \sigma_z = N\left(G_2 - G_1^2\right)^{\frac{1}{2}}\ , \tag{85}$$

where

$$G_1 = \prod_{k=1}^{L}\left\{\frac{k}{k - \frac{1}{N}}\right\}\ , \qquad G_2 = \prod_{k=1}^{L}\left\{\frac{k}{k - \frac{2}{N}}\right\}\ . \tag{86}$$

INPUT SIGNAL-TO-NOISE RATIO REQUIREMENTS

In this section, we will determine the CDFs of the outputs of the three nonlinear systems considered above, namely (31), (46), and (62) - (63), for the case where a signal is present in <u>one</u> of the input RVs. This will enable us to use the results given in (27) - (29) for determination of required input SNRs for a specified system detection probability $Pd_1 = 1-S_1$.

NORMALIZER

The nonlinear transformation of immediate interest is the normalizer given by (31). The PDF of denominator average RV a is given by (37), while the PDF of numerator RV x_0 with signal present will be modified from (4) to the form

$$p_x(u;R) = \frac{1}{1+R} \exp\left(\frac{-u}{1+R}\right) \quad \text{for } u > 0 \; , \tag{87}$$

where R is the input power SNR. The corresponding CDF is

$$C_x(X;R) = 1 - \exp\left(\frac{-X}{1+R}\right) \quad \text{for } X > 0 \; . \tag{88}$$

By reference to (33), (38), and (88), we find the CDF of RV y in (31) for signal present to be

$$C_y(Y;R) = \int_0^\infty dt \; \frac{M^M \, t^{M-1} \exp(-Mt)}{(M-1)!} \left[1 - \exp\left(\frac{-Yt}{1+R}\right)\right] =$$

$$= 1 - \left(1 + \frac{Y/M}{1+R}\right)^{-M} \quad \text{for } Y > 0 \; . \tag{89}$$

We now employ this result in (27) to obtain

$$S_1 = 1 - \left(1 + \frac{Y_t/M}{1+R_t}\right)^{-M} \quad \text{for } 1 \le t \le T . \tag{90}$$

The solution for the required input SNR R_t is then given by

$$R_t = \frac{Y_t/M}{(1 - S_1)^{-1/M} - 1} - 1 \quad \text{for } 1 \le t \le T . \tag{91}$$

We must repeat here the caution mentioned in (29), namely that specified probability S_1 at threshold Y_t must be less than or equal to probabilities $\{P_t\}$ in order that nonnegative SNRs $\{R_t\}$ result in (91). This is reasonable since it corresponds physically to demanding a larger detection probability when signal is present than when absent.

OR-ING DEVICE

Here, we are interested in the or-ing device characterized by (46) when signal is present in one of the RVs $\{x_m\}$. The CDF of RV y is then given by

$$C_y(Y;R) = C_x(Y;R)\ [C_x(Y)]^{M-1} =$$

$$= \left[1 - \exp\left(\frac{-Y}{1+R}\right)\right]\ [1 - \exp(-Y)]^{M-1} \quad \text{for } Y > 0\ , \qquad (92)$$

where we used (88) and (4).

If we now employ (92) in (27), we have to satisfy

$$S_1 = \left[1 - \exp\left(\frac{-Y_t}{1+R_t}\right)\right]\ \left[1 - \exp\left(-Y_t\right)\right]^{M-1} \quad \text{for } 1 \le t \le T\ . \qquad (93)$$

The solution for the required input SNR is given by

$$R_t = \frac{Y_t}{-\ln(1 - S_1/Q_t)} - 1 \quad \text{for } 1 \le t \le T\ , \qquad (94)$$

where

$$Q_t = \left[1 - \exp\left(-Y_t\right)\right]^{M-1} \quad \text{for } 1 \le t \le T\ . \qquad (95)$$

Again, (29) must be satisfied.

NORMALIZER AND OR-ING DEVICE

The nonlinear transformation to be investigated here is the combination of normalization and or-ing, as characterized by (62) and (63), when signal is present only in RV w_{10}. Therefore, only RV x_1 in (63) contains signal.

The CDFs of RVs $\{x_k\}$, for $2 \le k \le K$, are given by (65). On the other hand, the CDF for x_1 is available by reference to (89), in the form

$$C_x(X;R) = 1 - \left(1 + \frac{X/M}{1+R}\right)^{-M} \quad \text{for } X > 0 \ . \tag{96}$$

Therefore, the CDF for RV y in (63) is given by

$$C_y(Y;R) = C_x(Y;R) \; [C_x(Y)]^{K-1} =$$

$$= \left[1 - \left(1 + \frac{Y/M}{1+R}\right)^{-M}\right] \left[1 - \left(1 + \frac{Y}{M}\right)^{-M}\right]^{K-1} \quad \text{for } Y > 0 \ , \tag{97}$$

where we used (96) and (65).

When we equate this result to the specified probability S_1 according to (27), we obtain

$$S_1 = \left[1 - \left(1 + \frac{Y_t/M}{1+R_t}\right)^{-M}\right] \left[1 - \left(1 + \frac{Y_t}{M}\right)^{-M}\right]^{K-1} \quad \text{for } 1 \le t \le T \ . \tag{98}$$

The solution for the required input SNRs $\{R_t\}$ is given by

$$R_t = \frac{Y_t/M}{(1 - S_1/Q_t)^{-1/M} - 1} - 1 \quad \text{for } 1 \le t \le T \;, \tag{99}$$

where

$$Q_t = \left[1 - \left(1 + \frac{Y_t}{M}\right)^{-M}\right]^{K-1} \quad \text{for } 1 \le t \le T \;. \tag{100}$$

Restriction (29) must be satisfied here also.

Finally, if several detection probabilities Pd_1, Pd_2, Pd_3, are of interest, we must satisfy (30), where

$$\max\{1-P_t\} \le \min\{Pd_j\} \;. \tag{101}$$

As checks on the results in this subsection on combined normalization and or-ing, we observe for $M = \infty$ (that is, *no* normalization), (100) reduces to (95), where K here corresponds to M there for or-ing alone. Also, (99) reduces to (94). On the other hand, for finite M, but with $K = 1$ (that is, no or-ing), then (100) reduces to $Q_t = 1$, in which case (99) reduces to (91).

SUMMARY

We have determined the false alarm and detection probabilities for three different nonlinear transformations, namely a normalizer, an or-ing device, and a normalizer followed by or-ing. In particular, results are given for the following statistics of the outputs of each device: the PDF, the CDF, the inverse CDF, and either the moments or the cumulants (depending on their relative tractability). These results are sufficient to compute the receiver operating characteristics (ROCs) of the processors, if desired. However, we have also been able to solve explicitly for the input SNR required to realize specified false alarm and detection probabilities; this largely circumvents the need to compute ROCs.

Two programs, with numerical examples of their execution, are listed in appendix B. The first corresponds to the case where the number T of thresholds is kept fixed as the size of the nonlinear transformation is changed from M to N; see (20) - (23). On the other hand, the second program allows the number of thresholds to change from T to U as the size of the nonlinear transformation is changed from M to N; see (24) - (26).

Both programs are written for the general case where there is both normalization (of size M) and or-ing (of size K) included in the data processing; see (62) - (63). By making M infinite, the program will handle the case of or-ing alone; on the other hand, by setting K = 1, the program addresses the case of normalization alone. Thus, these two programs cover all the cases addressed in

this investigation. (Since it is not possible to actually set M infinite in a program, this situation is handled by setting M to any value less than or equal to 2, in order to flag this case in the program, and then branching appropriately at various points in the program. The substitute equations for this case of infinite M come, of course, from the earlier analysis for or-ing alone. For finite variance, normalization requires $M > 2$; see, for example, (77) or (79).)

We have not included the effects of averaging after the normalization and/or or-ing in this study. Hence, the required input SNRs calculated here sometimes turn out to be rather large. The exact analysis including averaging would be rather involved, since the new decision variable would have a characteristic function given by a power of the characteristic function $f_y(\xi)$ of current output variable y; that is, from (68),

$$f_y(\xi) = K \int_0^\infty du \, \exp(i\xi u) \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1} \left(1 + \frac{u}{M}\right)^{-M-1} . \quad (102)$$

This is probably best handled through the use of fast Fourier transforms. The integrand decays as u^{-M-1} for large u, which is attractive since M, the normalization size, is generally fairly large in order to guarantee decent performance.

In the meantime, in order to get a rather rough idea of the improvement attainable by employment of averaging, it is suggested that the rule of thumb [3; (C-10)] for the input signal-to-noise ratio improvement in dB, $-5 \log A$, be used, where A is the number of independent quantities averaged.

APPENDIX A. SIMPLIFICATION OF SUMS (76) AND (77)

Here, we will convert the alternating series in (76) and (77) into finite products that retain significance, even for large values of K. We begin with the first line of (52) and expand the power term in a binomial series, obtaining

$$f_y(\xi) = M \int_0^\infty du \, \exp(i\xi u - u) \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \exp(-ku) =$$

$$= M \sum_{k=0}^{M-1} \binom{M-1}{k} \frac{(-1)^k}{k + 1 - i\xi} \, . \qquad \text{(A-1)}$$

Now, equate (A-1) to its alternative expression in the last line of (52), and then replace M everywhere by K+1, getting

$$(K+1) \sum_{k=0}^{K} \binom{K}{k} \frac{(-1)^k}{k + 1 - i\xi} = \prod_{k=1}^{K+1} \left(\frac{k}{k - i\xi}\right) \, . \qquad \text{(A-2)}$$

Now let $z = 1 - i\xi$ in (A-2) and simplify; there follows

$$\sum_{k=0}^{K} \binom{K}{k} \frac{(-1)^k}{k + z} = \frac{K!}{(z)_{K+1}} \, . \qquad \text{(A-3)}$$

Thus, the alternating series has been converted into a finite product which is useful for all values of K without loss of significance.

The use of (A-3) on (76) yields the result quoted in (78) and (79) for the first moment $\mu_y(1)$. On the other hand, in order to convert (77), it is necessary to take the preliminary step of breaking the rational function into two parts according to

$$\frac{1}{(k-b)(k-a)} = \frac{1}{b-a}\left[\frac{1}{k-b} - \frac{1}{k-a}\right] , \tag{A-4}$$

and then to use (A-3) once with $z = -1/M$ and once with $z = -2/M$. After manipulation, simplification, and cancellation of common terms with the square of $\mu_y(1)$, the end result for the standard deviation of RV y is found to be just the second term in (78). The results in (78) - (79) have been numerically checked with the original defining integral (72), for $j = 1$ and $j = 2$, for several values of M and K; they agree within the accuracy of the computer employed.

A more general result than (A-3) is available by means of a different approach. First, for general values of a, note that

$$\binom{a}{k} = \frac{(-1)^k\,(-a)_k}{k!} , \quad \frac{1}{k+z} = \frac{1}{z}\,\frac{(z)_k}{(z+1)_k} . \tag{A-5}$$

Then, the following alternating sum can be manipulated into a more useful form, namely

$$\sum_{k=0}^{\infty} \binom{a}{k} \frac{(-1)^k}{k+z} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{(-a)_k\,(z)_k}{(z+1)_k\,k!} = \frac{1}{z}\,F(-a,z;z+1;1) = \frac{\Gamma(1+a)\,\Gamma(z)}{\Gamma(1+a+z)} , \tag{A-6}$$

where we used [2; 15.1.1 and 15.1.20]. If we set $a = K$ in (A-6), it reduces to (A-3). (A-6) is also equal to $B(1+a,z)$.

APPENDIX B. PROGRAMS

There are two programs listed in this appendix, both in BASIC for the Hewlett-Packard 9000 computer. None of the variables are declared INTEGER; thus, for example, input parameters M, K, N, L are all treated as REAL variables throughout.

The first program, listed on pages 42 - 44, requires that the number of thresholds T be maintained the same when the sizes, M and K, of the normalizer and or-ing device, respectively, are changed to N and L. On the other hand, the second program, listed on pages 46 - 49, allows the number of thresholds to change from T to U, which can be either larger or smaller.

The listings are heavily keyed to the explicit equations in the main text; this should enable the user to identify and modify particular manipulations if desired. It should be noted that, in the programs, the or-ing size begins at value K and gets changed to L. If these results are to be compared with the or-ing <u>only</u> results in the text, where the parameter M was used, it is necessary to make the switch from K in the program to M in the text. Example outputs from both programs are presented after the listing for each case.

```
10  ! NORMALIZER & ORING, EQUISPACED IN POWER, SAME NO. OF THRESHOLDS
20    P1=.85                    !  SPECIFIED PROBABILITY, (9)
30    Dby=10                    !  DECIBEL RATIO OF THRESHOLDS, (10)
40    T=7                       !  NUMBER OF THRESHOLDS, (10)
50    M=8                       !  INITIAL NORMALIZER SIZE > 2, (62)
60  ! FOR NO INITIAL NORMALIZATION, THAT IS, M INFINITE, SET M <= 2
70    K=12                      !  INITIAL OR-ING SIZE > 0, (63)
80    N=16                      !  NEW NORMALIZER SIZE > 2, (81)
90  ! FOR NO NEW NORMALIZATION, THAT IS, N INFINITE, SET N <= 2
100   L=24                      !  NEW OR-ING SIZE > 0, (82)
110   Pd1=.5                    !  SPECIFIED
120   Pd2=.9                    !   DETECTION
130   Pd3=.99                   !    PROBABILITIES, (30)
140   PRINT "P1 =";P1;"     Dby =";Dby;"       T =";T
150   PRINT "M =";M;"        K =";K;"       N =";N;"        L =";L
160   REDIM Y(1:T),Yb(1:T),P(1:T),Z(1:T),Zb(1:T)
170   DIM Y(99),Yb(99),P(99),Z(99),Zb(99)
180   Rk=1/K                    !  (67)
190   Rl=1/L                    !  (84)
200   IF M>2 THEN 300
210   F1=F2=0                   !  M INFINITE
220   FOR Ms=1 TO K             !  (56), m
230   R1=1/Ms
240   F1=F1+R1
250   F2=F2+R1*R1
260   NEXT Ms
270   Muy=F1                    !  (56)
280   Sigy=SQR(F2)              !  (56)
290   GOTO 390
300   R1=1/M                    !  M > 2
310   R2=2/M
320   F1=F2=1
330   FOR Ks=1 TO K             !  (79), k
340   F1=F1*Ks/(Ks-R1)          !  (79)
350   F2=F2*Ks/(Ks-R2)          !  (79)
360   NEXT Ks
370   Muy=M*(F1-1)              !  (78)
380   Sigy=M*SQR(F2-F1*F1)      !  (78)
390   IF N>2 THEN 490
400   G1=G2=0                   !  N INFINITE
410   FOR Ns=1 TO L             !  (61), n
420   R1=1/Ns
430   G1=G1+R1
440   G2=G2+R1*R1
450   NEXT Ns
460   Muz=G1                    !  (61)
470   Sigz=SQR(G2)              !  (61)
480   GOTO 580
490   R1=1/N                    !  N > 2
500   R2=2/N
510   G1=G2=1
520   FOR Ks=1 TO L             !  (86), k
530   G1=G1*Ks/(Ks-R1)          !  (86)
540   G2=G2*Ks/(Ks-R2)          !  (86)
550   NEXT Ks
560   Muz=N*(G1-1)              !  (85)
570   Sigz=N*SQR(G2-G1*G1)      !  (85)
580   PRINT "Muy =";Muy;"         Sigy =";Sigy
590   PRINT "Muz =";Muz;"         Sigz =";Sigz
600   PRINT
```

```
610  ! EQUISPACED IN DECIBELS: REMOVE 880-970 AND INSERT 620-710
620  ! R=1-P1^Rk              !  (9) & (67)
630  ! IF M>2 THEN 660
640  ! Y1=-LOG(R)             !  (58)
650  ! GOTO 670
660  ! Y1=M*(R^(-1/M)-1)      !  (67)
670  ! D1=10*LGT(Y1)          !  (14)
680  ! Deld=Dby/(T-1)         !  (15) & (16)
690  ! FOR Ts=1 TO T          !  (16), t
700  ! Dt=D1+Deld*(Ts-1)      !  (16)
710  ! Y(Ts)=Y=10^(Dt/10)     !  (17)
720  ! PRESET CONSTANTS: REMOVE 880-980 AND INSERT 730-760
730  ! DATA 1,3,5,7,9,10,11   !  USER MUST INPUT T NUMBERS
740  ! READ Yb(*)             !  NORMALIZED Y THRESHOLDS
750  ! FOR Ts=1 TO T          !  (18), t
760  ! Y(Ts)=Y=Muy+Sigy*Yb(Ts)!  (18)
770  ! PRESET PROBABILITIES: REMOVE 880-1030 AND INSERT 780-870
780  ! DATA .9,.99,.999,.9999,.99999,.999999,.9999999
790  ! READ P(*)              !  (19)
800  ! FOR Ts=1 TO T          !  (19), t
810  ! P=P(Ts)
820  ! R=1-P^Rk               !  (58)
830  ! IF M>2 THEN 860
840  ! Y(Ts)=Y=-LOG(R)        !  (58)
850  ! GOTO 870
860  ! Y(Ts)=Y=M*(R^(-1/M)-1) !  (9) & (67)
870  ! Yb(Ts)=(Y-Muy)/Sigy    !  (12)
880    R=1-P1^Rk              !  (9) & (67)
890    IF M>2 THEN 920
900    Y1=-LOG(R)             !  (58)
910    GOTO 930
920    Y1=M*(R^(-1/M)-1)      !  (67)
930    Ay=10^(Dby/10)         !  (10)
940    Ytc=Y1*Ay              !  (10)
950    Dely=(Ytc-Y1)/(T-1)    !  (11)
960    FOR Ts=1 TO T          !  (11), t
970    Y(Ts)=Y=Y1+Dely*(Ts-1) !  (11)
980    Yb(Ts)=(Y-Muy)/Sigy    !  (12)
990    IF M>2 THEN 1020
1000   R=EXP(-Y)              !  (57)
1010   GOTO 1030
1020   R=(1+Y/M)^(-M)         !  (13) & (66)
1030   P(Ts)=P=(1-R)^K        !  (66)
1040   Q=1-P^R1               !  (84)
1050   IF N>2 THEN 1080
1060   Z(Ts)=Z=-LOG(Q)        !  (21) & (60)
1070   GOTO 1090
1080   Z(Ts)=Z=N*(Q^(-1/N)-1) !  (21) & (84)
1090   Zb(Ts)=(Z-Muz)/Sigz    !  (22)
1100   NEXT Ts
1110   Dbz=10*LGT(Z(T)/Z(1))  !  (23)
1120   PRINT " Y THRESHOLDS         NORMALIZED          PROBABILITIES"
1130   FOR Ts=1 TO T
1140   PRINT Y(Ts),Yb(Ts),P(Ts)
1150   NEXT Ts
1160   PRINT
1170   PRINT " Z THRESHOLDS         NORMALIZED          PROBABILITIES"
1180   FOR Ts=1 TO T
1190   PRINT Z(Ts),Zb(Ts),P(Ts)
1200   NEXT Ts
1210   PRINT
1220   PRINT "Dbz =";Dbz
```

```
1230    PRINT
1240    PRINT " Pd1 =";Pd1;"          Pd2 =";Pd2;"          Pd3 =";Pd3
1250    PRINT
1260    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:"
1270    IF M>2 THEN 1400
1280    FOR Ts=1 TO T             !  (94), t
1290    Y=Y(Ts)
1300    Q=(1-EXP(-Y))^(K-1)       !  (95)
1310    D1=-LOG(1-(1-Pd1)/Q)      !  (94)
1320    Rt1=Y/D1-1                !  (94)
1330    D2=-LOG(1-(1-Pd2)/Q)
1340    Rt2=Y/D2-1
1350    D3=-LOG(1-(1-Pd3)/Q)
1360    Rt3=Y/D3-1
1370    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1380    NEXT Ts
1390    GOTO 1530
1400    Rm=-1/M                   !  (99)
1410    FOR Ts=1 TO T             !  (99), t
1420    Ym=Y(Ts)/M                !  (99) & (100)
1430    Q=(1+Ym)^(-M)             !  (100)
1440    Q=(1-Q)^(K-1)             !  (100)
1450    D1=(1-(1-Pd1)/Q)^Rm-1     !  (99)
1460    Rt1=Ym/D1-1               !  (99)
1470    D2=(1-(1-Pd2)/Q)^Rm-1
1480    Rt2=Ym/D2-1
1490    D3=(1-(1-Pd3)/Q)^Rm-1
1500    Rt3=Ym/D3-1
1510    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1520    NEXT Ts
1530    PRINT
1540    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:"
1550    IF N>2 THEN 1680
1560    FOR Ts=1 TO T             !  SIMILAR
1570    Z=Z(Ts)                   !  TO
1580    Q=(1-EXP(-Z))^(L-1)       !  (94)
1590    D1=-LOG(1-(1-Pd1)/Q)      !  &
1600    Rt1=Z/D1-1                !  (95)
1610    D2=-LOG(1-(1-Pd2)/Q)
1620    Rt2=Z/D2-1
1630    D3=-LOG(1-(1-Pd3)/Q)
1640    Rt3=Z/D3-1
1650    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1660    NEXT Ts
1670    GOTO 1810
1680    Rn=-1/N                   !  SIMILAR
1690    FOR Ts=1 TO T             !  TO
1700    Zn=Z(Ts)/N                !  (99)
1710    Q=(1+Zn)^(-N)             !  &
1720    Q=(1-Q)^(L-1)             !  (100)
1730    D1=(1-(1-Pd1)/Q)^Rn-1
1740    Rt1=Zn/D1-1
1750    D2=(1-(1-Pd2)/Q)^Rn-1
1760    Rt2=Zn/D2-1
1770    D3=(1-(1-Pd3)/Q)^Rn-1
1780    Rt3=Zn/D3-1
1790    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1800    NEXT Ts
1810    PRINT
1820    END
```

P1 = .85 Dby = 10 T = 7
M = 8 K = 12 N = 16 L = 24
Muy = 3.94631506068 Sigy = 2.09538979024
Muz = 4.32438211761 Sigz = 1.69328544323

Y THRESHOLDS	NORMALIZED	PROBABILITIES
5.7085601983	.841010653879	.85
14.2714004957	4.92752493268	.996679075641
22.8342407932	9.01403921147	.999753631023
31.3970810906	13.1005534903	.999965311792
39.9599213881	17.1870677691	.99999280763
48.5227616855	21.2735820479	.999998067526
57.085601983	25.3600963267	.999999374796

Z THRESHOLDS	NORMALIZED	PROBABILITIES
5.86721821227	.911149446675	.85
11.8779764405	4.46091021044	.996679075641
16.8024176865	7.36912705339	.999753631023
21.0784326281	9.89440414638	.999965311792
24.909916314	12.1571553566	.99999280763
28.4120734509	14.2254168838	.999998067526
31.6575625522	16.1420985125	.999999374796

Dbz = 7.32045232969

Pd1 = .5 Pd2 = .9 Pd3 = .99

SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:

7.18029480646	16.5239686654	26.8808804974
12.6997498763	21.2426978346	31.503743108
14.8466399857	23.3091924052	33.5584316626
16.269595645	24.6986419313	34.9428284925
17.3390310658	25.7492651765	35.9905902914
18.1963923488	26.5945353622	36.8340131363
18.9120679996	27.3017783991	37.5399641554

SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:

7.40601673703	16.6306689407	26.9721139815
11.9559811212	20.4527339701	30.7066466017
13.5624054665	21.9841310487	32.2269170012
14.587035072	22.975309045	33.2129962117
15.3357177535	23.704135924	33.9387696157
15.922698397	24.2777465869	34.5103143719
16.4038217386	24.749160188	34.9802228898

```
10   ! NORMALIZER & ORING, EQUISPACED IN POWER, DIFF. NO. OF THRESHOLDS
20     P1=.85                    !  SPECIFIED PROBABILITY, (9)
30     Dby=10                    !  DECIBEL RATIO OF THRESHOLDS, (10)
40     T=7                       !  INITIAL NUMBER OF THRESHOLDS, (10)
50     U=15                      !  NEW NUMBER OF THRESHOLDS, (24)
60     M=8                       !  INITIAL NORMALIZER SIZE > 2, (62)
70   ! FOR NO INITIAL NORMALIZATION, THAT IS, M INFINITE, SET M <= 2
80     K=12                      !  INITIAL OR-ING SIZE > 0, (63)
90     N=16                      !  NEW NORMALIZER SIZE > 2, (81)
100  ! FOR NO NEW NORMALIZATION, THAT IS, N INFINITE, SET N <= 2
110    L=24                      !  NEW OR-ING SIZE > 0, (82)
120    Pd1=.5                    !  SPECIFIED
130    Pd2=.9                    !   DETECTION
140    Pd3=.99                   !    PROBABILITIES, (30)
150    PRINT "P1 =";P1;"     Dby =";Dby;"       T =";T;"          U =";U
160    PRINT "M =";M;"        K =";K;"        N =";N;"          L =";L
170    REDIM Y(1:T),Yb(1:T),P(1:T),Zp(1:U),Zbp(1:U),Pp(1:U)
180    DIM Y(99),Yb(99),P(99),Zp(99),Zbp(99),Pp(99)
190    Rk=1/K                    !  (67)
200    Rl=1/L                    !  (84)
210    IF M>2 THEN 310
220    F1=F2=0                   !  M INFINITE
230    FOR Ms=1 TO K             !  (56), m
240    R1=1/Ms
250    F1=F1+R1
260    F2=F2+R1*R1
270    NEXT Ms
280    Muy=F1                    !  (56)
290    Sigy=SQR(F2)              !  (56)
300    GOTO 400
310    R1=1/M                    !  M > 2
320    R2=2/M
330    F1=F2=1
340    FOR Ks=1 TO K             !  (79), k
350    F1=F1*Ks/(Ks-R1)          !  (79)
360    F2=F2*Ks/(Ks-R2)          !  (79)
370    NEXT Ks
380    Muy=M*(F1-1)              !  (78)
390    Sigy=M*SQR(F2-F1*F1)      !  (78)
400    IF N>2 THEN 500
410    G1=G2=0                   !  N INFINITE
420    FOR Ns=1 TO L             !  (61), n
430    R1=1/Ns
440    G1=G1+R1
450    G2=G2+R1*R1
460    NEXT Ns
470    Muz=G1                    !  (61)
480    Sigz=SQR(G2)              !  (61)
490    GOTO 590
500    R1=1/N                    !  N > 2
510    R2=2/N
520    G1=G2=1
530    FOR Ks=1 TO L             !  (86), k
540    G1=G1*Ks/(Ks-R1)          !  (86)
550    G2=G2*Ks/(Ks-R2)          !  (86)
560    NEXT Ks
570    Muz=N*(G1-1)              !  (85)
580    Sigz=N*SQR(G2-G1*G1)      !  (85)
590    PRINT "Muy =";Muy;"          Sigy =";Sigy
600    PRINT "Muz =";Muz;"          Sigz =";Sigz
610    PRINT
```

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620  ! EQUISPACED IN DECIBELS: REMOVE 890-980 AND INSERT 630-720
630  ! R=1-P1^Rk                !  (9) & (67)
640  ! IF M>2 THEN 670
650  ! Y1=-LOG(R)               !  (58)
660  ! GOTO 680
670  ! Y1=M*(R^(-1/M)-1)        !  (67)
680  ! D1=10*LGT(Y1)            !  (14)
690  ! Deld=Dby/(T-1)           !  (15) & (16)
700  ! FOR Ts=1 TO T            !  (16), t
710  ! Dt=D1+Deld*(Ts-1)        !  (16)
720  ! Y(Ts)=Y=10^(Dt/10)       !  (17)
730  ! PRESET CONSTANTS: REMOVE 890-990 AND INSERT 740-770
740  ! DATA 1,3,5,7,9,10,11     !  USER MUST INPUT T NUMBERS
750  ! READ Yb(*)               !  NORMALIZED Y THRESHOLDS
760  ! FOR Ts=1 TO T            !  (18), t
770  ! Y(Ts)=Y=Muy+Sigy*Yb(Ts)! (18)
780  ! PRESET PROBABILITIES: REMOVE 890-1040 AND INSERT 790-880
790  ! DATA .9,.99,.999,.9999,.99999,.999999,.9999999
800  ! READ P(*)                !  (19)
810  ! FOR Ts=1 TO T            !  (19), t
820  ! P=P(Ts)
830  ! R=1-P^Rk                 !  (58)
840  ! IF M>2 THEN 870
850  ! Y(Ts)=Y=-LOG(R)          !  (58)
860  ! GOTO 880
870  ! Y(Ts)=Y=M*(R^(-1/M)-1)   !  (9) & (67)
880  ! Yb(Ts)=(Y-Muy)/Sigy      !  (12)
890    R=1-P1^Rk                !  (9) & (67)
900    IF M>2 THEN 930
910    Y1=-LOG(R)               !  (58)
920    GOTO 940
930    Y1=M*(R^(-1/M)-1)        !  (67)
940    Ay=10^(Dby/10)           !  (10)
950    Ytc=Y1*Ay                !  (10)
960    Dely=(Ytc-Y1)/(T-1)      !  (11)
970    FOR Ts=1 TO T            !  (11), t
980    Y(Ts)=Y=Y1+Dely*(Ts-1)   !  (11)
990    Yb(Ts)=(Y-Muy)/Sigy      !  (12)
1000   IF M>2 THEN 1030
1010   R=EXP(-Y)                !  (57)
1020   GOTO 1040
1030   R=(1+Y/M)^(-M)           !  (13) & (66)
1040   P(Ts)=(1-R)^K            !  (66)
1050   NEXT Ts
```

```
1060    Q1=1-P(1)^R1               !  (84)
1070    Qtc=1-P(T)^R1              !  (84)
1080    IF N>2 THEN 1120
1090    Z1=-LOG(Q1)                !  (60)
1100    Ztc=-LOG(Qtc)              !  (60)
1110    GOTO 1150
1120    Rn=-1/N                    !  (84)
1130    Z1=N*(Q1^Rn-1)             !  (21) & (84)
1140    Ztc=N*(Qtc^Rn-1)           !  (21) & (84)
1150    Delz=(Ztc-Z1)/(U-1)        !  (24)
1160    FOR Us=1 TO U              !  (24), u
1170    Zp(Us)=Z=Z1+Delz*(Us-1)!   (24)
1180    Zbp(Us)=(Z-Muz)/Sigz    !  (26)
1190    IF N>2 THEN 1220
1200    Pp(Us)=(1-EXP(-Z))^L    !  (25) & (59)
1210    GOTO 1230
1220    Pp(Us)=(1-(1+Z/N)^(-N))^L ! (25) & (83)
1230    NEXT Us
1240    Dbz=10*LGT(Zp(U)/Zp(1))!  (23)
1250    PRINT " Y THRESHOLDS          NORMALIZED           PROBABILITIES"
1260    FOR Ts=1 TO T
1270    PRINT Y(Ts),Yb(Ts),P(Ts)
1280    NEXT Ts
1290    PRINT
1300    PRINT " Z THRESHOLDS          NORMALIZED           PROBABILITIES"
1310    FOR Us=1 TO U
1320    PRINT Zp(Us),Zbp(Us),Pp(Us)
1330    NEXT Us
1340    PRINT
1350    PRINT "Dbz =";Dbz
1360    PRINT
1361      PAUSE
1370    PRINT   Pd1 =";Pd1;"          Pd2 =";Pd2;"          Pd3 =";Pd3
1380    PRINT
1390    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:"
1400    IF M>2 THEN 1530
1410    FOR Ts=1 TO T              !  (94), t
1420    Y=Y(Ts)
1430    Q=(1-EXP(-Y))^(K-1)        !  (95)
1440    D1=-LOG(1-(1-Pd1)/Q)       !  (94)
1450    Rt1=Y/D1-1                 !  (94)
1460    D2=-LOG(1-(1-Pd2)/Q)
1470    Rt2=Y/D2-1
1480    D3=-LOG(1-(1-Pd3)/Q)
1490    Rt3=Y/D3-1
1500    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1510    NEXT Ts
1520    GOTO 1660
```

```
1530    Rm=-1/M                 !  (99)
1540    FOR Ts=1 TO T           !  (99), t
1550    Ym=Y(Ts)/M              !  (99) & (100)
1560    Q=(1+Ym)^(-M)           !  (100)
1570    Q=(1-Q)^(K-1)           !  (100)
1580    D1=(1-(1-Pd1)/Q)^Rm-1   !  (99)
1590    Rt1=Ym/D1-1             !  (99)
1600    D2=(1-(1-Pd2)/Q)^Rm-1
1610    Rt2=Ym/D2-1
1620    D3=(1-(1-Pd3)/Q)^Rm-1
1630    Rt3=Ym/D3-1
1640    PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1650    NEXT Ts
1660    PRINT
1670    PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:"
1680    IF N>2 THEN 1810
1690    FOR Us=1 TO U           !  SIMILAR
1700    Z=Zp(Us)                !  TO
1710    Q=(1-EXP(-Z))^(L-1)     !  (94)
1720    D1=-LOG(1-(1-Pd1)/Q)    !  &
1730    R1=Z/D1-1               !  (95)
1740    D2=-LOG(1-(1-Pd2)/Q)
1750    R2=Z/D2-1
1760    D3=-LOG(1-(1-Pd3)/Q)
1770    R3=Z/D3-1
1780    PRINT 10*LGT(R1),10*LGT(R2),10*LGT(R3)
1790    NEXT Us
1800    GOTO 1940
1810    Rn=-1/N                 !  SIMILAR
1820    FOR Us=1 TO U           !  TO
1830    Zn=Zp(Us)/N             !  (99)
1840    Q=(1+Zn)^(-N)           !  &
1850    Q=(1-Q)^(L-1)           !  (100)
1860    D1=(1-(1-Pd1)/Q)^Rn-1
1870    R1=Zn/D1-1
1880    D2=(1-(1-Pd2)/Q)^Rn-1
1890    R2=Zn/D2-1
1900    D3=(1-(1-Pd3)/Q)^Rn-1
1910    R3=Zn/D3-1
1920    PRINT 10*LGT(R1),10*LGT(R2),10*LGT(R3)
1930    NEXT Us
1940    PRINT
1950    END
```

Pl = .85	Dby = 10	T = 7	U = 15
M = 8	K = 12	N = 16	L = 24
Muy = 3.94631506068		Sigy = 2.09538979024	
Muz = 4.32438211761		Sigz = 1.69328544323	

Y THRESHOLDS	NORMALIZED	PROBABILITIES
5.7085601983	.841010653879	.85
14.2714004957	4.92752493268	.996679075641
22.8342407932	9.01403921147	.999753631023
31.3970810906	13.1005534903	.999965311792
39.9599213881	17.1870677691	.99999280763
48.5227616855	21.2735820479	.999998067526
57.085601983	25.3600963267	.999999374796

Z THRESHOLDS	NORMALIZED	PROBABILITIES
5.86721821227	.911149446675	.85
7.70938566512	1.99907437995	.956529647638
9.55155311797	3.08699931323	.98667504332
11.3937205708	4.1749242465	.99560662072
13.2358880237	5.26284917978	.998447087595
15.0780554765	6.35077411306	.999415549265
16.9202229294	7.43869904633	.999767363279
18.7623903822	8.52662397961	.999902645376
20.6045578351	9.61454891288	.999957385298
22.4467252879	10.7024738462	.999980574274
24.2888927408	11.7903987794	.999990813211
26.1310601936	12.8783237127	.999995507461
27.9732276465	13.966248646	.999997734746
29.8153950993	15.0541735793	.999998825244
31.6575625522	16.14209851[illegible]5	.999999374796

Dbz = 7.32045232969

Pd1 = .5 Pd2 = .9 Pd3 = .99

SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:

7.18029480646	16.5239686654	26.8808804974
12.6997498763	21.2426978346	31.503743108
14.8466399857	23.3091924052	33.5584316626
16.269595645	24.6986419313	34.9428284925
17.3390310658	25.7492651765	35.9905902914
18.1963923488	26.5945353622	36.8340131363
18.9120679996	27.3017783991	37.5399641554

SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:

7.40601673703	16.6306689407	26.9721139815
9.64374630239	18.3703967446	28.6551266507
10.8741086655	19.4515609092	29.7168048237
11.7563696979	20.2655149832	30.5212118605
12.4659011251	20.9347216111	31.1845531717
13.0680388962	21.5092203112	31.7549390039
13.5941676551	22.0147264365	32.2573356581
14.0624498591	22.4668270369	32.7069765459
14.4847595558	22.8760203338	33.1141652155
14.8694755601	23.2498575248	33.4863332091
15.2228081965	23.5940083337	33.8290709181
15.5495188303	23.9128603228	34.1467104786
15.8533503185	24.2098871812	34.4426853709
16.1373031597	24.4878896233	34.7197662984
16.4038217386	24.749160188	34.9802228898

APPENDIX C. CORROBORATION OF PAGE 28

The CF corresponding to PDF $p_y(u)$ in (68) is

$$f_y(\xi) = K \int_0^\infty du \exp(i\xi u) \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1} \left(1 + \frac{u}{M}\right)^{-M-1} =$$

$$= K \exp(-i\xi M) \int_0^1 dx \,(1-x)^{K-1} \exp(i\xi M x^{-1/M}) \,, \qquad \text{(C-1)}$$

where we let $x = (1 + u/M)^{-M}$. Therefore

$$f_y(\xi) \exp(i\xi M) \sim K \sum_{j=0}^{M-1} \frac{(i\xi M)^j}{j!} \int_0^1 dx \,(1-x)^{K-1} x^{-j/M} \text{ as } \xi \to 0, \qquad \text{(C-2)}$$

where we expanded the second exponential in (C-1) in a power series, and have recognized that the integrals in (C-2) only converge for $j < M$. That is, (C-2) is an asymptotic expansion about $\xi = 0$, up through order $M-1$.

We now employ [1; 3.191 3 and 8.384 1] plus an extended version of the products in (79), namely

$$F_j = \prod_{k=1}^{K} \left\{\frac{k}{k - \frac{j}{M}}\right\} \text{ for } 0 \le j < M \,, \qquad \text{(C-3)}$$

to get, from (C-2),

$$f_y(\xi) \exp(i\xi M) \sim \sum_{j=0}^{M-1} \frac{(i\xi M)^j}{j!} F_j \text{ as } \xi \to 0 \,. \qquad \text{(C-4)}$$

Now let new RV q be defined as

$$q = \frac{y}{M} + 1 \ . \tag{C-5}$$

Then its CF is, upon use of (C-4),

$$f_q(\xi) = \overline{\exp(i\xi q)} = \overline{\exp(i\xi y/M + i\xi)} = f_y(\xi/M)\exp(i\xi) =$$

$$\sim \sum_{j=0}^{M-1} \frac{(i\xi)^j}{j!} F_j \quad \text{as } \xi \to 0 \ . \tag{C-6}$$

Therefore, the moments of RV q, up through order M-1, are directly just the products $\{F_j\}$ defined in (C-3), namely

$$\mu_q(j) = F_j \quad \text{for } 0 \le j < M \ . \tag{C-7}$$

The usual rules [4; page 70, (3.41)] for proceeding from moments $\{\mu_q(j)\}$ to cumulants $\{\chi_q(j)\}$ apply, at least for $j < M$.

From (C-5), since $y = M(q-1)$, the moments of RV y are

$$\mu_y(j) = \overline{y^j} = M^j\,\overline{(q-1)^j} = M^j \sum_{n=0}^{j} (-1)^n \binom{j}{n} \mu_q(j-n) \quad \text{for } j < M \ . \tag{C-8}$$

This result analytically confirms the conjecture on page 28. The cumulants for y follow readily from (C-5) as

$$\chi_y(1) = M[\chi_q(1) - 1] \ ,$$

$$\chi_y(j) = M^j\,\chi_q(j) \quad \text{for } 1 < j < M \ . \tag{C-9}$$

APPENDIX D. CHARACTERISTIC FUNCTION FOR PDF (68)

An asymptotic expansion for CF $f_y(\xi)$ was derived in (C-4). Here, we will derive a closed form for this CF in terms of exponential integrals.

AVERAGING A NONLINEAR TRANSFORMATION OF RV y

Suppose a positive RV y is subject to a nonlinear transformation, giving output g(y). The average of this latter quantity is

$$\overline{g(y)} = \int_0^\infty du\ g(u)\ p_y(u) \ , \qquad \text{(D-1)}$$

where $p_y(u)$ is the PDF of RV y. If we integrate by parts on (D-1), there follows

$$\overline{g(y)} = g(0) + \int_0^\infty du\ g'(u)\ [1 - C_y(u)] \ , \qquad \text{(D-2)}$$

assuming $C_y(0) = 0$ and

$$\lim_{u \to +\infty} g(u)\ [1 - C_y(u)] = 0 \ . \qquad \text{(D-3)}$$

Result (D-2) is an alternative to original definition (D-1); an example was utilized in (70).

An example of nonlinear transformation g(u) is

$$g(u) = \exp(i\xi u) \ , \quad g'(u) = i\xi \exp(i\xi u) \ , \qquad \text{(D-4)}$$

for which (D-1) and (D-2) become

$$f_y(\xi) = \overline{g(y)} = \overline{\exp(i\xi y)} = \int_0^\infty du \exp(i\xi u)\ p_y(u) = \tag{D-5}$$

$$= 1 + i\xi \int_0^\infty du \exp(i\xi u)\ [1 - C_y(u)] \ . \tag{D-6}$$

Requirement (D-3) is satisfied, since $C_y(u) \to 1$ as $u \to +\infty$.

DERIVATION OF CHARACTERISTIC FUNCTION

The CDF of the RV y of interest is given in (66). Substitution of (D-4) and (66) into (D-6) yields

$$f_y(\xi) = 1 + i\xi \int_0^\infty du \exp(i\xi u) \left\{1 - \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^K \right\} =$$

$$= 1 + i\xi \int_0^\infty du \exp(i\xi u) \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \left(1 + \frac{u}{M}\right)^{-Mk} =$$

$$= 1 + i\xi \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \int_0^\infty du \frac{\exp(i\xi u)}{(1 + u/M)^{Mk}} \ , \tag{D-7}$$

where we used (74).

Now define

$$\underline{E}_n(z) \equiv \exp(z)\ E_n(z) \quad \text{for } n \ge 1 \ , \tag{D-8}$$

where $E_n(z)$ is the exponential integral of order n [2; 5.1.4]. Also, let $x = 1 + u/M$ in (D-7), thereby leading to

$$f_y(\xi) = 1 + i\xi M \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \int_1^{\infty} dx \, \frac{\exp(i\xi Mx - i\xi M)}{x^{Mk}} =$$

$$= 1 + i\xi M \sum_{k=1}^{K} (-1)^{k-1} \binom{K}{k} \underline{E}_{Mk}(-i\xi M) \, . \qquad \text{(D-9)}$$

This closed form for the CF could utilize the recurrence relations [2; 5.1.14] for the exponential integrals, namely

$$\underline{E}_1(z) = \exp(z)\, E_1(z) \, ,$$

$$\underline{E}_n(z) = \frac{1}{n-1}\left[1 - z\, \underline{E}_{n-1}(z)\right] \quad \text{for } n \geq 2 \, . \qquad \text{(D-10)}$$

However, this recurrence looks troublesome for large z, due to its alternating character. Also, there appears to be a similar problem with alternating series (D-9) for large K. These latter results were not evaluated numerically.

If we start instead with PDF (68), the CF for y is given by

$$f_y(\xi) = K \int_0^{\infty} du \, \exp(i\xi u) \sum_{k=0}^{K-1} (-1)^k \binom{K-1}{k} \left(1 + \frac{u}{M}\right)^{-Mk-M-1} =$$

$$= KM \sum_{k=0}^{K-1} (-1)^k \binom{K-1}{k} \underline{E}_{Mk+M+1}(-i\xi M) \, . \qquad \text{(D-11)}$$

This is an alternative equivalent closed form to (D-9) for the CF of RV y.

A couple of special cases of the above results are listed below. For K = 1,

$$f_y(\xi) = M\,\underline{E}_{M+1}(-i\xi M) = 1 + i\xi M\,\underline{E}_M(-i\xi M)\ , \tag{D-12}$$

while for K = 2,

$$f_y(\xi) = 2M\left[\underline{E}_{M+1}(-i\xi M) - \underline{E}_{2M+1}(-i\xi M)\right]\ . \tag{D-13}$$

Finally, we have, directly from the PDF in (68), the CF of RV y in the form

$$f_y(\xi) = K\int_0^{\infty} du\ \exp(i\xi u)\ \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1}\ \left(1 + \frac{u}{M}\right)^{-M-1}\ . \tag{D-14}$$

Since the integrand decays as u^{-M-1} for large u, this is an attractive numerical form for M > 3. And since decent normalization is only attained for large M, form (D-14) is a very promising candidate for an FFT approach in these cases.

POST-AVERAGING OF RANDOM VARIABLE y

Suppose that independent RVs $\{y_j\}$ are subjected to further averaging according to

$$v = \sum_{j=1}^{J} y_j\ . \tag{D-15}$$

Then the CF of RV v is a power of the CF of RV y:

$$f_v(\xi) = [f_y(\xi)]^J \ . \tag{D-16}$$

This is the reason for the emphasis, above, placed on obtaining the CF of RV y.

However, suppose that the numerical evaluation of CF $f_y(\xi)$ is subject to some error; for example, the FFT evaluation of (D-14) would suffer aliasing in the ξ domain due to the necessity of sampling the integral on u. Therefore, let an estimate of $f_y(\xi)$ be available as

$$\tilde{f}_y(\xi) = f_y(\xi) + a(\xi) \ , \tag{D-17}$$

where $a(\xi)$ is an aliasing component. Then, the accompanying estimate of the CF of RV v is available according to

$$\tilde{f}_v(\xi) \equiv [\tilde{f}_y(\xi)]^J = [f_y(\xi) + a(\xi)]^J \cong f_v(\xi) + J\ f_y(\xi)^{J-1}\ a(\xi) \ , \tag{D-18}$$

where aliasing error $a(\xi)$ is presumed small. In particular, the estimate of the CF of RV v at the origin is

$$\tilde{f}_v(0) \cong 1 + J\ a(0) \ . \tag{D-19}$$

If the error in this estimate is desired less than 1E-15, for example, then the aliasing component at the origin must be bounded according to

$$|a(0)| < \frac{1}{J}\ 10^{-15} \ . \tag{D-20}$$

The larger J is in the average (D-15), the tighter is this bound in (D-20). Satisfaction of bound (D-20) will set the increment Δ_u to be used in approximating integral (D-14).

REFERENCES

[1] I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, Inc., New York, NY, 1980.

[2] **Handbook of Mathematical Functions**, U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, number 55, U. S. Government Printing Office, Washington, DC, June 1964.

[3] A. H. Nuttall and A. F. Magaraci, **Signal-to-Noise Ratios Required for Short-Term Narrowband Detection of Gaussian Processes**, NUSC Technical Report 4417, Naval Underwater Systems Center, New London, CT, 20 October 1972.

[4] M. G. Kendall and A. Stuart, **The Advanced Theory of Statistics, volume 1, Distribution Theory**, third edition, Hafner Publishing Company, New York, NY, 1969.

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Spectra and Covariances for "Classical" Nonlinear Signal Processing Problems Involving Class A Non-Gaussian Noise

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ABSTRACT

Because of the critical rôle of non-Gaussian noise processes in modern signal processing, which usually involves nonlinear operations, it is important to examine the effects of the latter on such noise and the extension to added signal inputs. Here, only non-Gaussian (specifically Class A) noise inputs, with an additive Gaussian component, are considered.

The "classical" problems of zero-memory nonlinear (ZMNL) devices serve to illustrate the approach and to provide a variety of useful output statistical quantities, e.g., mean or "dc" values, mean intensities, covariances, and their associated spectra. Here, Gaussian and non-Gaussian noise fields are introduced, and their respective temporal and spatial outputs are described and numerically evaluated for representative parameters of the noise and the ZMNL devices. Similar statistics for carriers which are phase or frequency modulated by Class A (and Gaussian) noise are also presented, numerically evaluated, and illustrated in the figures. A series of appendices and programs provide the technical support for the numerical analysis.

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PART II. MATHEMATICAL AND COMPUTATIONAL PROCEDURES

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LIST OF PRINCIPAL SYMBOLS

A	overlap index (A_A) of Class A noise
$\alpha(R,t)$	input (noise) field
B_ν	output covariance function, cf. (2.7a)
b	dimensionless factor, (2.44) et seq.
D_P, D_F	phase- and frequency-modulation scaling factors
δ	(Dirac) delta function
F_1, F_2	characteristic functions
FM	frequency modulation
$f(i\xi)$	Fourier transform of rectifier transfer function
g	ZMNL input-output transfer function
Γ'	σ_G^2/Ω_{2A} = "Gaussian" factor
$H_1^{(\nu)}$	scaling parameter for ν-th law detector outputs, (2.18a,b,c)
J_o	Bessel function of first kind
$K, K_{L,G}$	covariances
k_L, k_G	normalized covariances
ξ, ξ_1, ξ_2	characteristic function variables
$M_y, \hat{M}_y, \hat{M}'_y$	second-moment functions, (2.7), (2.8), (2.11)
$\mu_F, \hat{\mu}_F$	frequency-modulation indexes
$\mu_P, \hat{\mu}_P$	phase-modulation indexes
ν	power law of half-wave rectifier
$\hat{\nu}, \hat{\nu}$	normalized wavenumber, cf. (2.33a)
$\Omega_{G,A}$	covariance functions (angle-modulation), section 2.2-6

Ω_{2A}	intensity of Class A noise
Ω_o	spectral parameter in FM, PM, (2.42c)
PM	phase modulation
$\Delta R, \hat{\Delta R}$	spatial correlation distances
$\hat{R}$	array operator
Re z	real part of z
ρ	"overlap" correlation function, (2.3)
σ^2_{m+n}	noise variance
σ^2_G	intensity of Gauss noise component
T	observation time
T_s	waveform duration of elementary noise source
τ	$t_2 - t_1$, delay (correlation) time
$\hat{\tau}, \hat{\tau}'$	normalized delay parameters
$w(\hat{\omega})$	normalized intensity spectrum, (2.52)
$W_2, \hat{W}_2$	(wavenumber) intensity spectra
$W_{()}, \hat{W}_{()}$	(frequency) intensity spectra
x,x(t)	array outputs = inputs to ZMNL devices
y,y(t)	outputs of ZMNL devices
Y_a	correlation parameter, cf. (2.7b)
ZMNL	zero-memory nonlinear

LIST OF NORMALIZING AND NORMALIZED PARAMETERS

1. $\hat{\tau} = \beta\tau$; $\beta = 1/\overline{T}_s$; $\tau = t_2 - t_1$: correlation delay
   $\overline{T}_s$ = mean duration of typical interfering signal

2. $\rho = 1-|\hat{\tau}|$ for $|\hat{\tau}| < 1$, zero otherwise; (2.3): "overlap" correlation function

3. $\widehat{\Delta R} = \Delta R/\Delta_L$ (≥ 0); $\Delta R = |R_2 - R_1|$, correlation distance

4. $\hat{\omega} = \omega/\beta = \omega\overline{T}_s$; $\omega = 2\pi f$: normalized (angular) frequency

5. $\Delta\hat{\omega}_L = \Delta\omega_L/\beta$; normalized (frequency) spectrum bandwidth, for Class A noise model

6. $\Delta\hat{\omega}_G = \Delta\omega_G/\beta$; normalized (frequency) spectrum bandwidth, for Gauss noise component

7. Δ_L = rms spread of spatial covariance of non-Gaussian noise field component

8. Δ_G = rms spread of spatial covariance of Gaussian noise field component

9. $\hat{k} = k\Delta_L$: normalized wavenumber

10. $\Delta\omega_N$ = spectrum bandwidth of modulating (RC-Gauss) noise; cf. (2.43) and [1; section 14.1-3]; cf. (11) ff.

11. $\Delta\omega_N \rightarrow \Delta\omega_{N(L)}$ = bandwidth of non-Gaussian component of modulating noise in PM and FM;
    $\Delta\omega_N \rightarrow \Delta\omega_{N(G)}$ = bandwidth of Gaussian component of modulating noise in PM and FM

12. $\zeta = \tau\,\Delta\omega_N$, correlation variable, cf. (2.43) and (A.6-9) and figures 3.9a through 3.10b; see (A.6-9) for ζ_c

13. $\hat{\tau}' = \hat{\tau} - \beta(\Delta R/c_o)$, with $\hat{\tau} = \tau - \Delta R/c_o$, cf. (2.3a)

14. $\hat{\mu}_F = \mu_F(1+\Gamma')^{1/2}$: normalized FM index; (2.53)

15. $\hat{\mu}_P = \mu_P(1+\Gamma')^{1/2}$: normalized PM index; (2.53)

16. $\hat{\omega} = (\omega-\omega_o)/\Delta\omega_N$ = normalized displaced angular frequency, cf. (2.49)

SPECTRA AND COVARIANCES FOR "CLASSICAL" NONLINEAR SIGNAL PROCESSING PROBLEMS INVOLVING CLASS A NON-GAUSSIAN NOISE

PART I. ANALYTIC RESULTS AND NUMERICAL EXAMPLES

1. INTRODUCTION

Non-Gaussian noise fields play a critical rôle in modern signal processing because of the frequently dominant effects of such noise and interference in a wide variety of applications. Communication theory generally, and specifically telecommunications, electromagnetic and acoustic scattering, man-made and natural ambient noise, optics, and underwater acoustics, are common areas of interest in this respect. In the present report we are concerned primarily with underwater acoustic noise phenomena, but the models and results are canonical, that is, they take forms invariant to the particular physical application in question.

Specifically, we are concerned with various second-order statistics of non-Gaussian noise processes and fields after they have been subjected to different types of nonlinear operations, such as rectification and modulation. A generic problem here is the passage of non-Gaussian noise through a zero-memory nonlinear (ZMNL) device. The desired output statistics are typically the mean (dc), mean intensity (power), the covariance or correlation function, and the associated spectra. These last include wavenumber spectra in the case of noise fields, as well as the more general frequency-wavenumber spectra obtained by joint temporal and spatial Fourier transformations. Typical "classical" problems include: (i) rectification, (ii) determination of output spectra and covariances, (iii) calculation of (output) signal-to-noise ratios, (iv) modulation, (v) demodulation, and (vi) special systems, as for example, the spectrum analyzer. These and other problems involving ZMNL devices are described in detail in [1; chapters 5 and 12 - 17]. What is new here is the

use of the approximate <u>second-order</u> probability density functions and characteristic functions in the above applications when the noise processes are non-Gaussian.

A full treatment is given in a current study by Middleton, [2], which is an expanded version of his recent paper [3], which employs some of the results of the present report, namely, the calculated covariances and spectra. Here, we are content to summarize the pertinent analytic results, the corresponding examples of calculated covariances and spectra, and the various computational procedures associated with their evaluation. The details of the derivations are provided in [2] and [3]. Included here, also, is a selection of illustrations of the analytic results.

2. ANALYTICAL RESULTS: A SUMMARY

In the present study, we address three classical problems where the goals are the calculation of the covariance and associated intensity spectrum. Specifically, we consider:

<u>Problem I</u>. The half-wave ν-th law rectification of Class A noise fields and processes;

<u>Problem II</u>. Phase modulation of a carrier by a Class A noise process; and

<u>Problem III</u>. Frequency modulation of a carrier by a Class A noise process.

Class A noise, as noted in section 3 of [2], [3], is a canonical form of interference characterized by a coherent structure vis-à-vis the (linear) front-end stages of a typical receiver: negligible transients are produced at the output of these stages. Class B noise, on the other hand, is incoherent and highly impulsive, such that the front-end stages of the receiver generate an output which consists solely of (overlapping) transients. Here, the Class A models are tractable in the required second-order distribution and characteristic functions, whereas the Class B models are not and must

consequently be appropriately approximated in second-order; see [4] and [5] for additional information. In the present report, we shall consider examples of Class A noise only.

2.1 THE SECOND-ORDER CLASS A CHARACTERISTIC FUNCTION

In applications [1] - [3], the second-order characteristic function, $F_2(i\xi_1, i\xi_2)$, plays a key rôle: from it, we may obtain the aforementioned statistics of the outputs of ZMNL devices, spectra of angle-modulated carriers, and other usually second-order statistics of various nonlinear operations arising in a variety of communication and measurement operations. (See [2], [3] for further discussion.)

Here, we specifically use the approximate Class A noise characteristic function, F_2, including an additive Gaussian component, given by

$$\boxed{F_2(i\xi_1, i\xi_2)_{A+G} = \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1!\; m_2!} \times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \exp\left[-\frac{1}{2} Q^{(2)}_{m_1+n, m_2+n}(\xi_1, \xi_2)\right],} \tag{2.1}$$

where A ($=A_A$) is the "overlap" index, and where

$$Q^{(2)}_{m_1+n, m_2+n}(\xi_1, \xi_2) \equiv \xi_1^2\, \sigma^2_{m_1+n} + \xi_2^2\, \sigma^2_{m_2+n} + 2\xi_1\xi_2\, K^{(n)}_{L+G}, \tag{2.2a}$$

$$\sigma^2_{m+n} \equiv \left(\frac{m+n}{A} + \Gamma'_A\right) \Omega_{2A};\ \Omega_{2A} \quad \tfrac{1}{2}A\langle B_o^2\rangle = A\langle L^2\rangle;\ \Gamma'_A = \sigma_G^2/\Omega_{2A}; \tag{2.2b}$$

$$K^{(n)}_{L+G} = \left(n\, k_L/A + k_G\, \Gamma'_A\right) \Omega_{2A}; \tag{2.2c}$$

and k_L, k_G are the normalized covariances of the non-Gauss and Gauss components, respectively. Thus, $|k_{L,G}| \le 1$.

Here, ρ ($=\rho_A$) is the "overlap" correlation function

$$\rho(\tau') \equiv \begin{cases} 1 - \beta|\tau'| & \text{for } \beta|\tau'| \le 1 \\ 0 & \text{for } \beta|\tau'| > 1 \end{cases} , \quad \beta = 1/\overline{T}_s , \tag{2.3}$$

in which $\overline{T}_s$ is the mean duration of a typical noise-signal of intensity $\langle B_o^2\rangle/2 = \langle L^2\rangle$. The time delay τ' is given by

$$\tau' = \tau - \frac{\Delta R}{c_o} \quad \text{or} \quad \tau' = \tau \; (= t_2 - t_1) , \tag{2.3a}$$

respectively, for space-time fields and received temporal processes. The path delay $\Delta R/c_o$ ($= |R_2 - R_1|/c_o$) accounts for the time differential between propagation paths to the points at which processing occurs, cf. figure 2.1 ff, Case A. The quantities Ω_{2A} and σ_G^2 are, respectively, the intensity of the non-Gaussian and Gaussian components which constitute the general Class A model used here. (However, we note that the present Class A model belongs either to the strictly canonical Class A cases, where all interfering sources are equidistant from the observer, or more generally, to the much broader class of situations in which the effective source distribution is concentrated in an annulus whose inner-to-outer radii have a ratio O(1/2) or less. The former is exactly represented by (2.1) to second-order, while the latter is approximately so represented, albeit a good approximation as long as the aforementioned source annulus is not too large. See [5; section V, C], for example. For an exact treatment, see also [6], in an important class of physical models. Finally, differentiation of F_2, (2.1), in the usual way, gives us the (exact) covariance of the composite Class A and Gauss field, namely,

$$K_{A+G} = - \frac{\partial^2}{\partial\xi_1 \partial\xi_2} F_2 \bigg|_{\xi_1=\xi_2=0} = K_A + K_G , \tag{2.4}$$

which in normalized form is

$$k_{A+G}(\Delta R,\tau) = \frac{k_L + \Gamma' \, k_G}{1 + \Gamma'} \;. \tag{2.4a}$$

In practice, A is usually less than unity, say O(0.1 - 0.3) typically, so that only a comparatively few terms in $A^{m_1+m_2+n}$ are needed for numerical evaluation of (2.1) and the statistical quantities derived from it, cf. section 2.2 ff. Note that when $\beta|\tau| \geq 1$, $\rho = 0$, and $\Gamma' = 0$, we get

$$F_{2-A} = \left(e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \exp\left(-\tfrac{1}{2}\xi_1^2\sigma_m^2\right)\right) \left(e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \exp\left(-\tfrac{1}{2}\xi_2^2\sigma_n^2\right)\right)$$

$$= F_1(i\xi_1)_A \; F_1(i\xi_2)_A \;, \tag{2.4b}$$

as expected: there is now no correlation between process samples. With a Gaussian component, these will be correlated, of course, unless $|\tau| \to \infty$, so that $k_G \to 0$, cf. (2.2c).

2.2 PROBLEM I: HALF-WAVE ν-TH LAW RECTIFICATION (STATIONARY AND HOMOGENEOUS FIELDS)

Here we consider the problem of obtaining the second-order (second-moment) statistics, M_y, of a sampled noise field, $\alpha(R,t)$, after passage through a ZMNL device, g, when the noise is generally non-Gaussian. Various processing configurations are possible. We show two in figure 2.1, below. Analytically, we have, for stationary and homogeneous inputs [1; section 2.3-2]

$$M_y(\Delta R,\tau) = \overline{g(x_1)g(x_2)} = \frac{1}{(2\pi)^2} \int_{C_1}\int_{C_2} f(i\xi_1)\; f(i\xi_2)$$

$$\times \; F_2(i\xi_1, i\xi_2; \Delta R, \tau)_x \; d\xi_1 \; d\xi_2 = \overline{y_1 \; y_2} \;, \tag{2.5}$$

where $\Delta R = R_2 - R_1$, $\tau = t_2 - t_1$, and $f(i\xi)$ is the Fourier transform of the ZMNL device with $y_1 = y(R_1, t_1)$, etc. In the present cases, we have specifically

$$f(i\xi) = \beta\, \Gamma(\nu+1)/(i\xi)^{\nu+1}\ , \quad \nu > -1\ , \tag{2.6}$$

for these half-wave ν-th law rectifiers [1; (2.101a,b)].

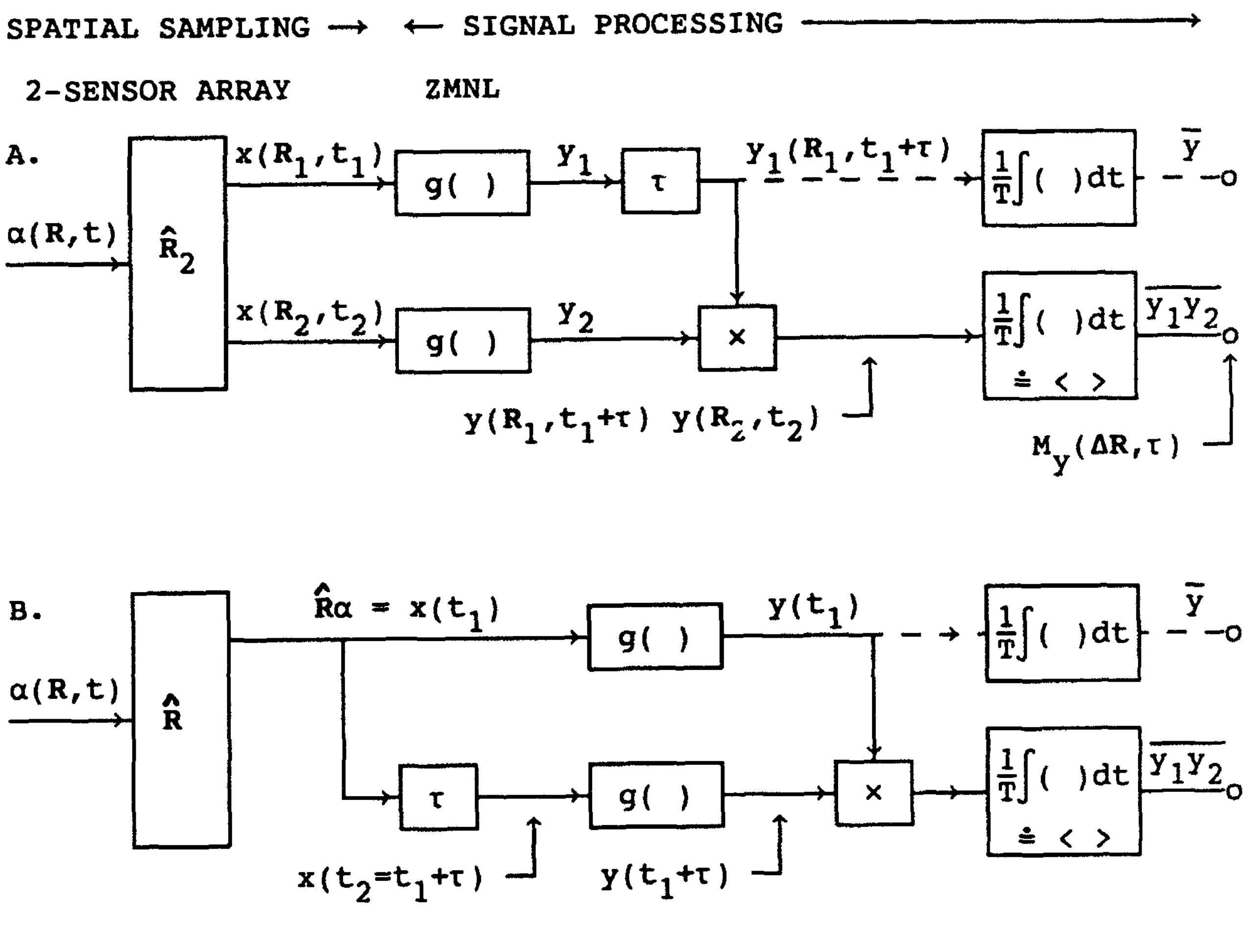

Figure 2.1 A. Two-point sensor array ($\hat{R}_2$) giving sampled field at two space-time points. B. A general array ($\hat{R}$) (preformed beam), converting the field $\alpha(R,t)$ into a single (time) process $x(t_1)$. Both are followed by ZMNL devices, delays, and averaging, as indicated schematically.

For the Class A non-Gaussian noise inputs of section 2.1 above, we find that the (normalized) second-moment M_y for the resulting rectified field is now

$$\boxed{M_y(\hat{\Delta R},\tau) = \exp[-A(2-\rho)] \sum_{m_1,m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1!\ m_2!} \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \times\left(\frac{n+m_1}{A} + \Gamma'\right)^{\nu/2} \left(\frac{n+m_2}{A} + \Gamma'\right)^{\nu/2} B_\nu\Big|_{m_1,m_2,n}\ ,} \quad (2.7)$$

where we have further postulated the noise field to be isotropic, $\Delta R \to |\Delta R|$, and where specifically,

$$B_\nu\Big|_a \equiv B_\nu(Y|m_1,m_2,n) = \Gamma^2\left(\frac{\nu+1}{2}\right)\ {}_2F_1\left(-\frac{\nu}{2},-\frac{\nu}{2};\frac{1}{2};Y_a^2\right)$$

$$+\ 2Y_a\ \Gamma^2\left(\frac{\nu}{2}+1\right)\ {}_2F_1\left(\frac{1-\nu}{2},\frac{1-\nu}{2};\frac{3}{2};Y_a^2\right)\ , \quad (2.7a)$$

$$Y_a \equiv \frac{\frac{n}{A}\,k_L + \Gamma'\ k_G}{\left(\frac{m_1+n}{A} + \Gamma'\right)^{\frac{1}{2}} \left(\frac{m_2+n}{A} + \Gamma'\right)^{\frac{1}{2}}}\ ;\quad a = (m_1,m_2,n)\ ,\quad |Y_a| \le 1\ . \quad (2.7b)$$

Specifically, also, we have the following normalized forms

$$\hat{M}_y \equiv M_y/\Omega_{2A}^{\nu}2^{\nu}/4\pi\ ;\quad \hat{\tau}' \equiv \beta\tau'\ ,\quad \beta = 1/\overline{T}_s\ ,\quad \text{cf. } (2.3)\ ,$$

$$\hat{\Delta R} \equiv \Delta R/\Delta_L\ ,\quad \Delta_L = \text{correlation distance},\quad \Delta R = |R_2-R_1|\ . \quad (2.8)$$

For numerical results, we select the following models for the space-time covariance functions of the isotropic and stationary non-Gaussian and Gaussian components of the input noise field:

$$k_L \equiv \exp\left(-\Delta R^2/\Delta_L^2 - \tfrac{1}{2}(\Delta\omega_L\tau'/\beta)^2\right) = \exp\left(-\hat{\Delta R}^2 - \tfrac{1}{2}(\Delta\hat{\omega}_L\hat{\tau}')^2\right)\ ; \quad (2.9a)$$

$$k_G \equiv \exp\left(-\Delta R^2/\Delta_G^2 - \tfrac{1}{2}(\Delta\omega_G\tau'/\beta)^2\right) = \exp\left(-\hat{\Delta R}^2(\Delta_L/\Delta_G)^2 - \tfrac{1}{2}(\Delta\hat{\omega}_G\hat{\tau}')^2\right)\ ;$$

$$\Delta\hat{\omega}_L \equiv \Delta\omega_L/\beta \ , \quad \Delta\hat{\omega}_G \equiv \Delta\omega_G/\beta \ . \tag{2.9b}$$

Here, Δ_G is a correlation distance, and $\Delta\omega_L$, $\Delta\omega_G$ are angular frequency spreads associated with the respective non-Gaussian and Gaussian components of the input field. Note that if we define the correlation distance Δ_L as that where $k_L = 1/e$ ($\hat{\tau}' = 0$), then $\Delta_L = \Delta R_L$, etc.

For the special cases of ν considered here, we also observe (from [1; (A.1-39)]) that B_ν may be expressed in closed form:

$$B_0(Y) = \pi + 2 \arcsin(Y) \ , \tag{2.10a}$$

$$B_1(Y) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} + \frac{\pi}{2}Y \ , \tag{2.10b}$$

$$B_2(Y) = \left(\frac{1}{2} + Y^2\right)\left(\frac{\pi}{2} + \arcsin(Y)\right) + \frac{3}{2}Y\left(1 - Y^2\right)^{\frac{1}{2}} \ . \tag{2.10c}$$

2.2-1 GAUSS PROCESSES ALONE (A=0)

When only a Gauss noise field is originally present, that is, A = 0, for example, $\Omega_{2A} = 0$, (2.7) reduces to the classical result [1; page 541, (13.4a)]:

$$\boxed{\hat{M}'_y \equiv \hat{M}_y\Big|_{A=0} = B_\nu\Big|_{a=0} \ ; \quad M_y = \frac{(2\psi)^\nu}{4\pi}\hat{M}_y \ ; \quad Y_a \to Y_o = k_G .} \tag{2.11}$$

For comparison with the non-Gaussian cases (A>0), we choose to have equal <u>input</u> noise intensities. This means that

$$\psi_{A=0} = \sigma_G^2 + \Omega_{2A} = \Omega_{2A}(1 + \Gamma') \ ,$$

so that

$$\hat{M}_y\Big|_{A=0} = (1 + \Gamma')^\nu \, B_\nu\Big|_{a=0} \ , \quad Y_o = k_G \ , \tag{2.12}$$

and $\hat{M}'_y$ is then to be compared with $\hat{M}_y$, A > 0. When Γ' is small, as is usually the case, we can often replace $(1 + \Gamma')^\nu$ by unity.

At this point, following figure 2.1, we distinguish two classes of operation: (A), where a pair of point sensors is used to sample the noise field and we wish to consider both the space and temporal correlations of the sampled field at the two points (R_1,t_1), (R_2,t_2); and (B), where the space-time field is converted into a random process, x(t), by the beamforming array $(\hat{R})$, with an associated directionality embodied in the resultant beam (vide [7; sections IV B and VI A]).

2.2-2 CASE B, FIGURE 2.1

Let us consider the simpler case (Case B) of the time process first, cf. (B). For this, we set $\Delta R = 0$ formally in (2.7) et seq. above, since $x = \hat{R}\,\alpha(R,t)$ here and $\tau' = \tau = t_2 - t_1$, cf. (2.3a). See also [3; (3.2) et seq. and (3.11a)]. Then our ad hoc illustrative models of the process covariances k_L, k_G, are, from (2.9a,b), at once*

$$k_L = k_L(\tau) = \exp\left(-\tfrac{1}{2}(\Delta\omega_L\hat{\tau}/\beta)^2\right) = \exp\left(-\tfrac{1}{2}(\Delta\hat{\omega}_L\hat{\tau})^2\right) , \quad (2.13a)$$

$$k_G = k_G(\tau) = \exp\left(-\tfrac{1}{2}(\Delta\omega_G\hat{\tau}/\beta)^2\right) = \exp\left(-\tfrac{1}{2}(\Delta\hat{\omega}_G\hat{\tau})^2\right) . \quad (2.13b)$$

Accordingly, (2.7) reduces to

$$\text{Case B:}\quad M_y(0,\hat{\tau}) \equiv M_y(\hat{\tau})_B = (2.7),\ \text{with } Y_a = (2.7b),$$

$$\text{and (2.13a,b) and } \widehat{\Delta R} = 0 \text{ therein.} \quad (2.14)$$

We note that when $|\hat{\tau}| \geq 1$, $\rho = 0$, and $\hat{M}_y(0,|\hat{\tau}| \geq 1)$ reduces to a simpler relation [vis-à-vis (2.7)], viz.:

---

* A physically derived model of k_G and k_L may be made from [3; (3.11a)] with $L = \hat{R}\,L$, $\hat{R} = (2.9)$ etc., where L is typically given by [3; (3.3)], for example.

$$\boxed{\hat{M}_y(\hat{\tau})_B = e^{-2A} \sum_{m_1,m_2=0}^{\infty} \frac{A^{m_1+m_2}}{m_1!\; m_2!} \left(\frac{m_1}{A} + \Gamma'\right)^{\nu/2} \left(\frac{m_2}{A} + \Gamma'\right)^{\nu/2} B_\nu\Big|_a ,} \tag{2.14a}$$

where (2.7b) becomes

$$Y_a = \frac{\Gamma' \; k_G}{\left(\frac{m_1}{A} + \Gamma'\right)^{1/2} \left(\frac{m_2}{A} + \Gamma'\right)^{1/2}} , \qquad \rho = 0 , \tag{2.14b}$$

in $B_\nu\big|_a$.

Special cases of interest are:

I. THE INTENSITY $E(y^2)$: $\hat{\tau} = 0$, $\rho = 1$, $m_1 = m_2 = 0$, and (2.7), (2.14) reduce to

$$\boxed{\overline{y^2}\Big|_{norm} = \hat{M}_y(0)_B = \hat{M}_y(0,0) = B_\nu\Big|_{a=n} = e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma'\right)^{\nu} ,} \tag{2.15}$$

where now $Y_{a=n} = 1$, e.g., $k_L(0) = 1$ etc., and B_ν is independent of n, for example, for $Y_a = 1$,

$$B_\nu\Big|_{a=n} = \begin{Bmatrix} 2\pi & \text{for } \nu = 0 \\ \pi & \text{for } \nu = 1 \\ 3\pi/2 & \text{for } \nu = 2 \end{Bmatrix} , \text{ cf. (2.10)} . \tag{2.16a}$$

For general ν, $Y_a = 1$, we have (from [1; (A.1-34)])

$$B_\nu\Big|_{a=n} = 2\pi^{1/2}\Gamma(\nu+\tfrac{1}{2}) , \quad \nu \geq 0 . \tag{2.16b}$$

Thus, (2.15) becomes

$$\overline{y^2}\Big|_{norm} = \hat{M}_y(0)_B = \hat{M}_y(0,0) = 2\pi^{1/2}\Gamma(\nu+\tfrac{1}{2}) \; e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma'\right)^{\nu} . \tag{2.17}$$

The unnormalized form is, from (2.8),

$$\overline{y^2} = M_y(0)_B = M_y(0,0) = \frac{2^{\nu-1}}{\pi^{1/2}}\,\Omega_{2A}^{\nu}\,\Gamma(\nu+\tfrac{1}{2})\;H_1^{(\nu)}(A,\Gamma') \qquad (2.18)$$

with

$$H_1^{(\nu)}(A,\Gamma') \equiv e^{-A}\sum_{n=0}^{\infty}\frac{A^n}{n!}\left(\frac{n}{A}+\Gamma'\right)^{\nu}$$

$$= \begin{cases} 1 & \text{for } \nu = 0\,, \quad (2.18a)\\ 1+\Gamma' & \text{for } \nu = 1\,, \quad (2.18b)\\ 1/A + (1+\Gamma')^2 & \text{for } \nu = 2\,. \quad (2.18c)\end{cases}$$

For other values of ν (>0), we must evaluate $H_1^{(\nu)}$ numerically.

II. THE MEAN VALUE, $\bar{y}$; $|\hat{t}| \to \infty$

Now $\rho = 0$, $n = 0$, $Y_a = 0$, and (2.7) reduces directly, upon use of (2.18), to

$$\boxed{\begin{aligned}\bar{y}^2\Big|_{norm} = \hat{M}_y(\infty)_B = \hat{M}_y(0,\infty) &= \Gamma^2\left(\frac{\nu+1}{2}\right)\left\{e^{-A}\sum_{m=0}^{\infty}\frac{A^m}{m!}\left(\frac{m}{A}+\Gamma'\right)^{\nu/2}\right\}^2\\ &= \Gamma^2\left(\frac{\nu+1}{2}\right)\;H_1^{(\nu/2)}(A,\Gamma')^2\,.\end{aligned}} \qquad (2.19)$$

The unnormalized form of (2.19) is, from (2.8),

$$\bar{y}^2 = M_y(\infty)_B = M_y(0,\infty) = \frac{2^{\nu}\,\Omega_{2A}^{\nu}}{4\pi}\,\Gamma^2\left(\frac{\nu+1}{2}\right)\;H_1^{(\nu/2)}(A,\Gamma')^2\,, \qquad (2.20)$$

and for ν even, we find, from (2.18a,b,c)

$$H_1^{(0)} = 1\,,\quad H_1^{(1)} = 1+\Gamma'\,,\quad H_1^{(2)} = \frac{1}{A} + (1+\Gamma')^2\,. \qquad (2.21)$$

III. THE CONTINUUM INTENSITY: $\overline{y^2} - \bar{y}^2$

From (2.18) and (2.20) we get at once the general result for $\nu \geq 0$,

$$\boxed{P_c \equiv \overline{y^2} - \bar{y}^2 = 2^\nu \, \Omega_{2A}^\nu \left\{ \frac{\Gamma(\nu+\frac{1}{2})}{2\pi^{\frac{1}{2}}} H_1^{(\nu)} - \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{2\pi^{\frac{1}{2}}} H_1^{(\nu/2)} \right)^2 \right\},} \quad (2.22)$$

which is the generalization of [1; (13.7)], in the classical purely Gaussian cases, to the present, dominant non-Gaussian noise component Ω_{2A} ($\gg \sigma_G^2$). In these classical cases, we can show at once that

$$\lim_{\Omega_{2A}\to 0} H_1^{(\nu)} \Omega_{2A}^\nu \to \lim_{\Omega\to 0} e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n\Omega}{A} + \sigma_G^2 \right)^\nu \to \sigma_G^{2\nu} \; (= \psi^\nu) \; , \quad (2.23)$$

where $\Omega_{2A} \to \infty$ implies $A \to 0$ and $\overline{B_0^2} \to 0$, cf. (2.2b), so that (2.22) becomes, as expected,

$$P_c\Big|_{Gauss} = 2^\nu \, \sigma_G^{2\nu} \left\{ \frac{\Gamma(\nu+\frac{1}{2})}{2\pi^{\frac{1}{2}}} - \frac{\Gamma^2\left(\frac{\nu+1}{2}\right)}{4\pi} \right\} \quad (> 0) \; , \quad \nu \geq 0 \; . \quad (2.24)$$

Figure 13.5 of [1] shows (2.24) as a function of rectifier law (ν), as well as (2.18), (2.20) in these Gaussian cases. In the present, more general, situation of Class A noise, the results are more complex, as expected, with now <u>two</u> additional parameters (A,Γ'), descriptive of this much broader class of interference.

2.2-3: CASE A, FIGURE 2.1

We turn now to the more general problem of the covariance of the Class A non-Gaussian random field, sampled according to procedure (A), shown schematically in figure 2.1 earlier. Here, $x = \alpha(R,t)$, sensed at (R_1,t_1), (R_2,t_2), where $L = L$, cf. (3.3) in [3; (3.2)]. Equation (2.7) applies here, with $\Delta R \neq 0$ (as well as for $\Delta R = 0$), and we use (2.9a,b) for our illustrative examples, which are discussed in section 3 following. At this point, we recall from (2.3a) that the proper time delay to use is $\tau' = \tau - \Delta R/c_o$ in $\rho = \rho(\tau')$, and in some of the structural elements of the noise field covariances, cf. [3; (3.11b,c)].

CASE I: $\hat{\tau}' = 0$

From (2.7), we have $\rho = 1$, $m_1 = m_2 = 1$, giving

$$\hat{M}_Y(\widehat{\Delta R},0) = e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma'\right)^{\nu} B_{\nu}\Big|_{a=n}, \quad \rho = 1, \quad \therefore \tau = \frac{\Delta R}{c_o}, \tag{2.25}$$

where (2.7b) is specifically

$$Y_{a=n} = \frac{\frac{n}{A}\, k_L(\widehat{\Delta R},0) + \Gamma'\, k_G(\widehat{\Delta R},0)}{\frac{n}{A} + \Gamma'} . \tag{2.25a}$$

For calculations, (2.9a,b) are used, with B_{ν} given by (2.7a), where (2.25a) provides Y_a. When $\widehat{\Delta R} = 0$, (2.25) reduces to (2.15) et seq. for the total intensity of the field observed at $R_1 = R_2$.

CASE II: $\hat{\Delta R} \to \infty$, $|\tau'| > 1$

When $\hat{\Delta R} \to \infty$, we obtain different results, depending on τ'. Here $\rho = 0$, $Y_a \to 0$, cf. (2.9a,b) in (2.7b), and therefore $n = 0$. Accordingly, (2.7) becomes

$$\hat{M}_y(\infty, |\tau'| > 1) = \hat{M}_y(\infty,\infty) = \hat{M}_y(0,\infty) = \bar{y}^2_{norm}\ , \ (2.19)\ . \qquad (2.26)$$

The fact that $\hat{\Delta R} \to \infty$ ensures that $Y_a \to 0$, a behavior similar to that for Case (B) above, when we consider the purely Gaussian noise process, section 2.2-1.

CASE III: $\hat{\Delta R} \to \infty$, $0 < |\tau'| < 1$

Here, $\rho > 0$ while $Y_a \to 0$, so that B_ν, (2.7b), becomes $\Gamma^2(\nu+\frac{1}{2})$ once more. The second moment function (2.7) is now

$$\boxed{\begin{aligned} M_y(\hat{\Delta R},\tau') &= \Gamma^2\left(\frac{\nu+1}{2}\right) \exp[-A(2-\rho)] \sum_{m_1,m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1!\ m_2!} \\ &\times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \left(\frac{n+m_1}{A} + \Gamma'\right)^{\nu/2} \left(\frac{n+m_2}{A} + \Gamma'\right)^{\nu/2}\ , \end{aligned}} \qquad (2.27)$$

which is a minor simplification of (2.7).

CASE IV: $\hat{\Delta R} \to \infty$, $|\tau'| = 0$

In this special situation, where $\tau = \Delta R/c_o \to \infty$ in such a way that $\tau' = 0$ and therefore $\rho = 1$, $Y_a = 0$, we obtain directly from (2.27) the comparatively simple result,

$$\boxed{\hat{M}_y(\infty,0) = \Gamma^2\left(\frac{\nu+1}{2}\right) H_1^{(\nu)} \quad (> 0)\ .} \qquad (2.28)$$

2.2-4: REMARKS

At first glance, as $\Delta R \to \infty$, we might expect M_y always to reduce to $\bar{y}^2$, e.g., $K_y \equiv M_y - \bar{y}^2 = 0$ for the covariance of the rectified space-time field. This is expectedly the case for the covariance (and second-moment) function of the input Class A and Gauss noise field components $\alpha(R,t)$, as we can see directly from (2.9a,b), or from [3; (3.11b,c)] for example, in the physically derived cases. However, the process or field $y = g(x)$ here is the result of a nonlinear operation, cf. (2.5), (2.6), which severely distorts the input waveform and generates all kinds of modulation products, associated with the spatial as well as the temporal variations of the input field. This accounts for the departures in Cases III, IV of $M_y(\infty,\hat{\tau}')$ from $\bar{y}^2$, while certainly $M_x(\infty,\tau') \to \bar{x}^2 = 0$, (since $\bar{x} = 0$ initially here).

From the various limiting results above, we see that

$$M_y(0,0) > M_y(\infty,0) \quad \text{and} \quad M_y(0,0) > M_y(0,\infty) \;, \qquad (2.29a)$$

and

$$M_y(\infty,0) \gtreqless M_y(0,\infty) \quad \text{depending on } A,\ \Gamma',\ \text{and } \nu \;, \qquad (2.29b)$$

with

$$M_y(0,0) - M_y(0,\infty) > 0 \;, \quad \text{cf. (2.19) and (2,20)} \;, \qquad (2.29c)$$

$$M_y(\infty,\infty) = M_y(0,\infty) = \bar{y}^2 \;, \text{ cf. (2.20) and (2.26)} \;, \qquad (2.29d)$$

whereas

$$M_x(0,0) > M_x(0,\infty) = M_x(\infty,0) = \bar{x}^2 = 0 \;. \qquad (2.29e)$$

Finally, we note that (2.11), (2.12) apply here, also, for the Gauss-alone cases, where now

$$\Omega_{2A}(1 + \Gamma') \to \sigma_G^2 \quad \text{and} \quad B_\nu\Big|_{a=0} \to (1 + \Gamma')^\nu \, B_\nu\Big|_a = 0 \;.$$

2.2-5: SPECTRA

The various intensity spectra associated with the output of the processor (cf. figure 2.1) are important also, as they show how the energy in this output is distributed. Here, we consider two types of spectra, respectively, for the rectified spatial field (A) and for the process (B), namely the wavenumber and the frequency spectrum of $y(R,0)$ and $y(0,t)$. In particular, wavenumber spectra are useful in the analysis of spatially distributed phenomena, paralleling the analysis of time-dependent phenomena.

I: WAVENUMBER SPECTRUM

The wavenumber intensity spectrum is defined here by

$$W_2(k,0)_y = W_2(k,\tau)_y\Big|_{\tau=0} \equiv \iint_{\Delta R} M_y(\Delta R,0)\ \exp(ik\cdot\Delta R)\ d(\Delta R) \qquad (2.30a)$$

$$= 2\pi \int_0^\infty M_y(\Delta R,0)\ J_o(k\Delta R)\ \Delta R\ d(\Delta R) = W_2(k,0)_y\ , \qquad (2.30b)$$

with

$$\mathbf{k} = (k_x,k_y)\ ,\quad \Delta R = |\Delta \mathbf{R}|\ ,\quad k = |\mathbf{k}| \qquad (2.30c)$$

for these <u>isotropic fields</u>, where **k** is an (angular) vector wavenumber. Using the normalization of (2.8), we get, with $\hat{k} \equiv k\Delta_L$,

$$\boxed{\hat{W}_2(\hat{k},0)_y \equiv \frac{W_2(k,0)_y}{\Delta_L^2\ 2^\nu\ \Omega^{2\nu}/4\pi} = 2\pi \int_0^\infty \hat{M}_y(x,0)\ J_o(\hat{k}x)\ x\ dx} \qquad (2.31)$$

for the normalized wavenumber intensity spectrum.

Since $\hat{M}_y(\infty,0)$ is nonvanishing, cf. (2.28), there is a dc component, or δ-function, in the general wavenumber spectrum. We will use the relations

$$\int_0^\infty x\ J_o(\hat{k}x)\ dx = \frac{1}{\hat{k}}\ \delta(\hat{k} - 0)\ , \quad \hat{k} = \left(\hat{k}_x^2 + \hat{k}_y^2\right)^{\frac{1}{2}} = |\hat{k}|\ , \quad \hat{k} = (\hat{k},\phi)\ , \tag{2.32}$$

where we must remember that $\hat{k}$ is two-dimensional. With ν a vector wavenumber defined by

$$\hat{k} = 2\pi\hat{\nu}\ [= (\hat{\nu},\phi)]\ , \quad \hat{k} = 2\pi\hat{\nu} = 2\pi|\hat{\nu}|\ , \tag{2.33a}$$

and using the relation

$$\delta(ax - b) = \frac{1}{a}\ \delta\left(x - \frac{b}{a}\right) \quad \text{for } a > 0\ , \tag{2.33b}$$

we also show that

$$\delta(\hat{k}_x-0)\ \delta(\hat{k}_y-0) = \frac{1}{2\pi\hat{k}}\ \delta(\hat{k}-0) = \frac{1}{(2\pi)^3\hat{\nu}}\ \delta(\hat{\nu}-0)$$

$$= \frac{1}{(2\pi)^2}\ \delta(\hat{\nu}_x-0)\ \delta(\hat{\nu}_y-0)\ , \quad \hat{\nu} = |\hat{\nu}|\ , \quad \hat{\nu} = \left(\hat{\nu}_x^2 + \hat{\nu}_y^2\right)^{\frac{1}{2}}\ . \tag{2.34}$$

Applying (2.32) - (2.34), with

$$\hat{W}_2(\hat{k},\phi)_y = 2\pi \int_0^\infty x\ J_o(\hat{k}x)\ \left[\hat{M}_y(x,0) - \hat{M}_y(\infty,0)\right]\ dx$$

$$+ 2\pi\ \hat{M}_y(\infty,0) \int_0^\infty x\ J_o(\hat{k}x)\ dx \tag{2.35a}$$

$$\boxed{\begin{aligned} &= \hat{W}_2(\hat{k},0)_{y\text{-cont}} + (2\pi)^2\ \hat{M}_y(\infty,0)\ \delta(\hat{k}_x-0)\ \delta(\hat{k}_y-0) \\ &= \hat{W}_2(\hat{k},0)_{y\text{-cont}} + \hat{M}_y(\infty,0)\ \delta(\hat{\nu}_x-0)\ \delta(\hat{\nu}_y-0)\ , \end{aligned}} \tag{2.35b}$$

which defines $\hat{W}_2(\hat{k},0)_{y\text{-cont}}$, the continuous portion of the spectrum and shows the dc term in $\hat{k}$- or $\hat{\nu}$-space, as convenient. It is $\hat{W}_{2\text{-cont}}$ with which we are concerned in the specific numerical examples of section 3 ff.

II. FREQUENCY SPECTRUM

Here we employ the Wiener-Khintchine theorem [1; (3.42)] to write for the frequency spectrum of y

$$W_y(f) = 2\int_{-\infty}^{\infty} M_y(0,\tau)\exp(-i\omega\tau)\,d\tau = B_o\int_0^{\infty}\hat{M}_y(0,\hat{t})\cos(\hat{\omega}\hat{t})\,d\hat{t}, \quad (2.36)$$

where

$$B_o \equiv \Omega_{2A}^{\nu}\,2^{\nu}/\pi\beta;\quad \hat{t} = \beta\tau;\quad \omega = 2\pi f;\quad \hat{\omega} \equiv \omega/\beta;\ \therefore\ \hat{f} = f/\beta. \quad (2.36a)$$

Accordingly, we define the normalized frequency intensity spectrum of y as

$$\boxed{\hat{W}_y(\hat{f}) \equiv W_y(f)/B_o = \int_0^{\infty}\hat{M}_y(0,\hat{t})\cos(\hat{\omega}\hat{t})\,d\hat{t}\ .} \quad (2.37)$$

Again, there is a dc component, since $\hat{M}_y(0,\infty) = \bar{y}^2$ (> 0), cf. (2.20). We have

$$\hat{W}_y(f) = \int_0^{\infty}\left[\hat{M}_y(0,\hat{t})-\hat{M}_y(0,\infty)\right]\cos(\hat{\omega}\hat{t})\,d\hat{t} + \frac{1}{2}\hat{M}_y(0,\infty)\,\delta(\hat{f}-0), \quad (2.38)$$

since

$$\int_0^{\infty}\cos(\omega x)\,dx = \pi\,\delta(\omega-0) = \frac{1}{2}\,\delta(f-0)\ .$$

As in the wavenumber cases above (Case I), we are concerned with the continuous part of the spectrum, viz.

$$\boxed{\hat{W}_y(f)_{cont} = \int_0^\infty \left[\hat{M}_y(0,\hat{\tau}) - \bar{y}^2\right] \cos(\omega\hat{\tau})\, d\hat{\tau} ,} \tag{2.39}$$

which is also illustrated numerically in section 3 ff.

III. WAVENUMBER FREQUENCY SPECTRUM

The wavenumber frequency spectrum is defined by

$$W_2(k,\omega)_y = \iint_{-\infty}^{\infty} M_y(\Delta R,\tau)\, \exp(ik\cdot\Delta R - i\omega\tau)\, d(\Delta R)\, d\tau , \tag{2.40}$$

with $\omega = 2\pi f$. The associated wavenumber spectrum $W_2(k,0)$ used in (2.30) is obtained from $W_2(k,\tau)\Big|_{\tau=0}$. In normaliz form, we have for (2.40), in these isotropic cases,

$$\hat{W}_2(\hat{k},\hat{\omega})_y = \left(2^\nu\, \Omega_{2A}^\nu\, \Delta_L^2/(4\pi\beta)\right)^{-1} W_2(k,\omega)_y$$

$$= \iint_{-\infty}^{\infty} \hat{M}_y(\widehat{\Delta R},\hat{\tau})\, \exp(i\hat{k}\cdot\widehat{\Delta R} - i\hat{\omega}\hat{\tau})\, d(\widehat{\Delta R})\, d\hat{\tau} ;$$

$$\boxed{\hat{W}_2(\hat{k},\hat{\omega})_y = 2\pi \int_0^\infty \int_0^\infty \hat{M}_y(x,\hat{\tau})\, J_o(\hat{k}x)\, \exp(-i\hat{\omega}\hat{\tau})\, x\, dx\, d\hat{\tau} .} \tag{2.41}$$

The various dc components are readily extracted, as in Cases I and II above. Numerical examples of this joint intensity spectrum are reserved to a possible subsequent study. The results of section 3 show the marginal spectra (Cases I, II) of this more general situation.

2.2-6: FREQUENCY AND PHASE MODULATION BY CLASS A AND GAUSSIAN NOISE

This is a Case (B) situation, cf. figure 2.1, where $\Delta R = 0$ and we are concerned only with the received (non-Gaussian) noise process which is used to angle-modulate a (high frequency) carrier f_o. For the analysis, see [3; section II].

The general result for the covariance of the carrier modulated by Class A and Gauss noise is found to be

$$\boxed{K_y(\tau)_{A+G} = \tfrac{1}{2}A_o^2 \ \mathrm{Re}\left[\exp\left(i\omega_o\tau - D_o^2\Omega(\tau)_G - A(2-\rho) + 2A(1-\rho)\ \exp\left[-D_o^2\Omega_o/A\right]\right)\right]}\ , \tag{2.42}$$

where now, cf. [1; (4.2),(14.14c)],

$$\Omega(\tau)_{\substack{G\\A}}\Big|_{FM} = \left(\sigma_G^2 \text{ or } \tfrac{1}{A}\Omega_{2A}\right) \int_0^{|\tau|} (|\tau|-\lambda)\ k(\lambda)\ d\lambda \qquad (D_o = D_F)$$

$$= \left(1 \text{ or } \tfrac{1}{A}\right) \int_0^\infty W_{(G \text{ or } L)}(f)\ \frac{1-\cos(\omega\tau)}{\omega^2}\ df\ . \tag{2.42a}$$

Also, cf. [1; (14.2), (14.14c)],

$$\Omega(\tau)_{\substack{G\\A}}\Big|_{PM} = \left(\sigma_G^2 \text{ or } \tfrac{1}{A}\Omega_{2A}\right) \{k(0) - k(\tau)\}_{(G \text{ or } L)} \qquad (D_o = D_P) \tag{2.42b}$$

and

$$\Omega_o \to \Omega_o\Big|_{FM} = \int_0^\infty W_A(f)\ df/\omega^2 \quad \text{or} \quad \Omega_o\Big|_{PM} = \Omega_{2A}\ . \tag{2.42c}$$

For our numerical examples, we use the RC-spectrum of [1; section 14.1-3], where now

$$k_L(\zeta) = \exp(-b|\zeta|) \;, \quad k_G(\zeta) = \exp(-|\zeta|) \;, \quad \zeta = \tau\,\Delta\omega_N \;, \tag{2.43}$$

and therefore

$$\text{FM:}\quad D_F^2\,\Omega(\tau)_A = \frac{1}{Ab^2}\left(\mu_F^2\right)_A [\exp(-b|\zeta|) + b|\zeta| - 1] \;;$$

$$D_F^2\,\Omega(\tau)_G = \Gamma'\left(\mu_F^2\right)_A [\exp(-|\zeta|) + |\zeta| - 1] \;;$$

$$\left(\mu_F^2\right)_A = \Omega_{2A}\,D_F^2/\Delta\omega_N^2 \;; \tag{2.44a}$$

$$\text{PM:}\quad D_P^2\,\Omega(\tau)_A = \frac{1}{A}\left(\mu_P^2\right)_A [1 - \exp(-b|\zeta|)] \;;$$

$$D_P^2\,\Omega(\tau)_G = \Gamma'\left(\mu_P^2\right)_A [1 - \exp(-b|\zeta|)] \;;$$

$$\left(\mu_P^2\right)_A = D_P^2\,\Omega_{2A} \;, \tag{2.44b}$$

with b (> 0) a dimensionless quantity, as is ζ. The quantity $\Delta\omega_N$ is the bandwidth of the modulating (Gauss) noise, cf. (2.43). Note, also, that

$$\Gamma'\left(\mu_F^2\right)_A = \sigma_G^2\,D_F^2/\Delta\omega_N^2 = \left(\mu_F^2\right)_G \;; \quad \Gamma'\left(\mu_P^2\right)_A = \sigma_G^2\,D_P^2 = \left(\mu_P^2\right)_G \;. \tag{2.45}$$

The quantities $(\mu_{F,P}^2)_{()}$ are the respective modulation indexes for FM and PM, cf. [1; chapter 14].

Finally, we have for ρ in (2.3), now with $\Delta R = 0$,

$$\rho(\tau) \to \rho(\zeta) = \begin{Bmatrix} 1 - \dfrac{\beta|\zeta|}{\Delta\omega_N} & \text{for} & \dfrac{\beta|\zeta|}{\Delta\omega_N} \le 1 \\ 0 & \text{for} & \dfrac{\beta|\zeta|}{\Delta\omega_N} > 1 \end{Bmatrix} . \tag{2.46}$$

Putting the above (2.43), (2.44) in (2.42), we now specialize our results,

$$K_y(\tau)_{A+G} = \tfrac{1}{2}A_o^2\, k_o(\tau)\cos(\omega_o\tau)\ , \quad \text{with}\quad k_o(0) = 1\ , \tag{2.47}$$

to the normalized covariance $k_o(\tau)$, respectively, for FM and PM, and their associated spectra. We have for these carriers modulated by a sum of Gaussian and Class A noise:

I. FREQUENCY MODULATION

$$\boxed{\begin{aligned} k_o(\zeta)_{FM} = \exp\Big[&-\Gamma'\left(\mu_F^2\right)_A [\exp(-|\zeta|) + |\zeta| - 1] - A(2-\rho) \\ &+ A\rho\,\exp\left(-\frac{1}{Ab^2}\left(\mu_F^2\right)_A [\exp(-b|\zeta|) + b|\zeta| - 1]\right)\Big]\ , \end{aligned}} \tag{2.48}$$

with $\rho(\tau)$ given by (2.46). Here, $\Omega_o\big|_{FM} \to \infty$ in (2.42). Since

$$\lim_{\zeta\to\infty} k_o(\zeta)_{FM} = 0\ ,$$

there is no dc in k_{o-FM}, and hence all the original carrier power $(\sim A_o^2/2)$ is distributed into the sideband continuum for this highly nonlinear modulation, as expected [1; section 14.1-2].

The associated intensity spectrum for $k_o\big|_{FM}$ is defined by

$$W(\hat{\omega})_{A+G}\Big|_{FM} = \int_0^\infty k_o(\zeta)_{FM}\cos(\hat{\omega}\zeta)\, d\zeta\ , \qquad \hat{\omega} \equiv \frac{\omega-\omega_o}{\Delta\omega_N}\ , \tag{2.49}$$

which is determined by a direct cosine transform of $k_o(\zeta)_{FM}$. See appendix A.6 ff.

II. PHASE MODULATION

$$\boxed{\begin{aligned} k_o(\zeta)_{PM} = \exp\Big[&-\Gamma'\left(\mu_P^2\right)_A [1 - \exp(-|\zeta|)] + 2A(1-\rho)\exp\left(-\tfrac{1}{A}\left(\mu_P^2\right)_A\right) \\ &- A(2-\rho) + A\rho\exp\left(-\tfrac{1}{A}\left(\mu_P^2\right)_A [1 - \exp(-b|\zeta|)]\right)\Big] \,, \end{aligned}} \tag{2.50}$$

with $\rho(\zeta)$ again given by (2.46). We note that

$$k_o(0)_{PM} = 1 \,, \tag{2.51a}$$

as before; that is, the total (normalized) intensity is unity. Also

$$\boxed{k_o(\infty)_{PM} = \exp\left[-\Gamma'\left(\mu_P^2\right)_A - 2A\left(1 - \exp\left(-\tfrac{1}{A}\left(\mu_P^2\right)_A\right)\right)\right] \,:} \tag{2.51b}$$

this is the fraction of the power remaining in the carrier, so that

$$k_o(0)_{PM} - k_o(\infty)_{PM} = 1 - (2.51b) \,, \tag{2.51c}$$

which is the fraction of the power distributed in the sideband continuum.

The associated intensity spectrum of the sideband continuum is determined from

$$w(\hat{\omega})_{A+G}\Big|_{PM\text{-cont}} = \int_0^\infty \left[k_o(\zeta)_{PM} - k_o(\infty)_{PM}\right] \cos(\hat{\omega}\zeta)\, d\zeta \,. \tag{2.52}$$

See section 3 ff. for examples and appendix A.5 for the evaluation methods.

Finally, in the <u>equivalent Gaussian cases</u> (Gauss noise modulation of equal intensity and basic spectrum, e.g.,

$$\Gamma'\left(\mu_F^2\right)_A \to \Gamma'\left(\hat{\mu}_F^2\right)_A = \Gamma'\ \mu_{FA}^2\ (1+\Gamma') \quad \text{and} \quad k_G \to k_L\ ,$$

we see that (2.48), (2.50) reduce to

$$k_o(\zeta)_{FM\text{-}Gauss} = \exp\left[-\left(\hat{\mu}_F^2\right)_A \Gamma'\ [\exp(-b|\zeta|) + b|\zeta| - 1]/b^2\right]\ ,$$

$$\left(\hat{\mu}_F^2\right)_A = (1+\Gamma')\left(\mu_F^2\right)_A\ ; \tag{2.53a}$$

$$k_o(\zeta)_{PM\text{-}Gauss} = \exp\left[-\left(\hat{\mu}_P^2\right)_A \Gamma'\ [1 - \exp(-b|\zeta|)]\right]\ ,$$

$$\left(\hat{\mu}_P^2\right)_A = (1+\Gamma')\left(\mu_P^2\right)_A\ , \tag{2.53b}$$

with spectra obtained as before, from (2.49) and (2.52).

3. NUMERICAL ILLUSTRATIONS AND DISCUSSION

It is convenient to discuss the general results, namely the effects of (ZMNL) nonlinear rectifiers on, and modulation by, a mixture of Gaussian and non-Gaussian noise processes and fields, from the specific numerical calculations presented here in figures 3.1 - 3.10. These constitute a representative selection from the universe of possible parameter states [cf. "Summary of Normalized Parameters" and section 2, preceding]. This is done here on a per-figure basis, as noted below. In each case, the dc component is removed: only the covariance or continuous spectrum is calculated. We recall that there are two cases to distinguish: Case A, $\tau'=\tau-\Delta R/c_o$, a 2-element array; and Case B, $\tau'=\tau=t_2-t_1$, a preformed beam. See figure 2.1 and (2.3a).

All spectra shown here are normalized to have area (under the spectrum level) of unity, i.e., the spectral normalization is obtained by dividing the spectrum by the value of the associated covariance at its origin. The normalization of the covariances themselves is obtained by dividing by the value at $\hat{\tau}=0$ or $\hat{\Delta R}=0$.

I. GAUSS NOISE ALONE

FIGURE 3.1

This figure shows the normalized temporal covariance ($\hat{\Delta R}=0$) for both the input and output of a ZMNL half-wave ν-th law ($\nu\geq0$) detector, when $\nu = 0,1,2$ and when only Gaussian noise ($A=0$) is applied to these nonlinear devices. These curves are based on (2.11) with (2.7a), where $Y_a = k_G$, (2.96), with $\Delta\hat{\omega}_G \equiv \Delta\omega_G/\beta = 5$ here. The normalization is with respect to the covariance maximum; e.g., the normalized covariance shown in figure 3.1 is obtained from $[(2.11)/(2.11)\hat{\tau}=0]$, $\hat{\Delta R}=0$. These results apply for both cases A,B of figure 2.1, where now $\hat{\tau}'=\hat{\tau}$, since $\hat{\Delta R}=0$, cf. (2.3a) and remarks.

As expected (cf. [1; chapter 13]), the general nonlinearity (2.6), $\nu \geq 0$, contracts the covariance, which is equivalent to spreading the spectrum vis-à-vis the input , cf. figure 3.2, below. Moreover, the greater the distortion (ν=0,2), usually the greater are these effects. [See appendix A.1.]

FIGURE 3.2

This is the same situation as shown in figure 3.1, except that the normalized intensity (frequency) spectrum is calculated now [cf. section 2.2-5, Case II, (2.39)] with $\hat{M}_y(0,\hat{t})$, (2.11), used in (2.39). Observe the greatly broadened spectra, particularly at the low spectral levels, where the greater spread occurs for the "super-clipper", ν=0, cf. remarks, figure 3.1; also, appendix A.3.

FIGURE 3.3

For the same purely Gaussian field above, cf. (2.11) and (2.96), with $\hat{t},\hat{t}'=0$, the spatial covariance is calculated, with parameters $\Delta_L/\Delta_G = 5^{\frac{1}{2}}$, using (2.11) as before. The normalization is with respect to the covariance at $\hat{\Delta R}=0$. Again, one observes the same kind of contraction in the covariance as noted in figure 3.1. [See appendix A.2.]

FIGURE 3.4

This is the wavenumber analogue of the frequency spectrum of figure 3.2, now with $\hat{t}',\hat{t}=0$, and is obtained from (2.35a,b) with $\Delta_L/\Delta_G = 5^{\frac{1}{2}}$. The rectification operation similarly spreads the wavenumber spectrum, with the greatest distortion (ν=0) yielding the greatest wavenumber spread, as expected from the corresponding contraction of the associated covariance, cf. figure 3.3 above. [See appendix A.4.]

CLASS A PLUS GAUSS NOISE

FIGURE 3.5

The temporal covariance here is given by the general result (2.7), with the associated relations (2.7), (2.8), (2.9), wherein $\widehat{\Delta R}=0$, so that $\hat{\tau}=\hat{\tau}'=t_2-t_1$, as before, and where $B_\nu|_a$, (2.7a), is now given analytically by (2.10) for $\nu = 0,1,2$. Here, the parameter values are $\Delta\hat{\omega}_G \equiv \Delta\omega_G/\beta = 5^{1/2}$, as before, now with $A=0.2$, $\Gamma_A'=10^{-3}$, $\Delta\omega L/\beta \equiv \Delta\hat{\omega}_L = 1$ for the Class A non-Gaussian noise component, typically.

Again, for the super-clipper ($\nu=0$), the contraction in the normalized covariance is greatest, cf. figure 3.1. But the contribution of the comparatively strong non-Gaussian component exaggerates this effect. [See appendix A.1.]

FIGURE 3.6

The corresponding intensity (frequency) spectrum ($\widehat{\Delta R}=0$), obtained from (2.7) in (2.39), however, shows a fine-structure not exhibited when Gauss noise alone ($A=0$) is applied to these ZMNL devices. The spectral levels for the case $\nu=0$, ($A=0$) and ($A>0$), cf. figure 3.2 with figure 3.6, are approximately the same, whereas the other inputs, cases $\nu=1,2$, are much elevated as f becomes larger, again due to the presence of the structured Class A noise, when $\beta\overline{T}_s \leq 1$, cf. (2.3): on the average, the original Class A "signals" are of comparatively short duration, or spectrally wide to begin with, so that clipping further spreads the spectrum. [See appendix A.3.]

FIGURE 3.7

The spatial covariance when Class A noise is added to the Gaussian input shows analogous behavior, cf. figures 3.3 and 3.5: the covariance is compressed vis-à-vis the input, but more so than in the Gauss-alone situations. Again, (2.7) - (2.10) are employed. [See appendix A.2.]

FIGURE 3.8

The corresponding wavenumber (intensity) spectrum with Class A noise and the Gaussian component, obtained from (2.7) - (2.10) in (2.39), is shown here. Comparison with figure 3.4 indicates a broader spectral input, due to the non-Gaussian component, but a relatively narrower output, although the latter is still noticeably spread vis-à-vis the original input. [See appendix A.4.]

FIGURE 3.9

Finally, we consider the angle-modulation cases described in section 2.2-6 above, where weak to strong angle modulations ($\mu \sim 1$ to 50) by Class A noise, with a weak ($\Gamma'=10^{-3}$) Gaussian modulation component, is employed.

For phase modulation by non-Gaussian noise, based on (2.50) with (2.44b), (2.45), (2.46), the resulting normalized intensity (frequency) spectra are obtained by applying (2.50) to (2.52), where $f = \hat{\omega}/2\pi$; $\hat{\omega} = (\omega-\omega_o)/\Delta\omega_N$, cf. (2.49). Note the "spike" at $f \sim 0.1$, followed by a variety of sidelobes which rise as the phase modulation index μ_P increases. The spike is now bounded at $f \approx 0.8$, at the -10 dB level, when $\mu_P = 50$. As expected, the larger indexes (μ_P) produce broader spectra. [See appendix A.5.]

FIGURE 3.10

For frequency modulation by non-Gaussian noise, from (2.48) with (2.49) and (2.44a), the corresponding intensity (frequency) spectra again exhibit a continuous spike ($f < 0.1$). With small modulation indexes (μ_F), the spectra are less broad than for the larger indexes, as expected. The non-Gaussian noise component dominates the spectrum here. [See appendix A.6.]

EXTENSIONS

Other situations where the second-order Class A probability density functions may be applied are noted in [2] and [3]. We list some of the extensions of the analysis to the following "classical" problems:

1) The inclusion of representative signals, with Gauss and non-Gauss (Class A) noise, in the problems already treated here (section 2);
2) The case of the full-wave square-law rectifier, with both Class A and B noise, as well as Gauss noise;
3) The extension of 2) to include general broadband and narrowband signals;
4) The calculation of signal-to-noise ratios and deflection criteria, cf. [1; section 5.3-4].
5) Covariances and spectra for ZMNL system outputs, with signals as well as non-Gaussian noise inputs;
6) The rôle of the electromagnetic (or acoustic) interference (EMI or AcI) scenario, cf. [5; section 2B,5];
7) Evaluation of the large (FM,PM) indexes, or asymptotically Gaussian cases, cf. [12].

Further opportunities to extend the classical theory [2],[3], now with non-Gaussian noise inputs, are evident from the examples and methods described in [1; chapters 5, 12 - 16], for instance.

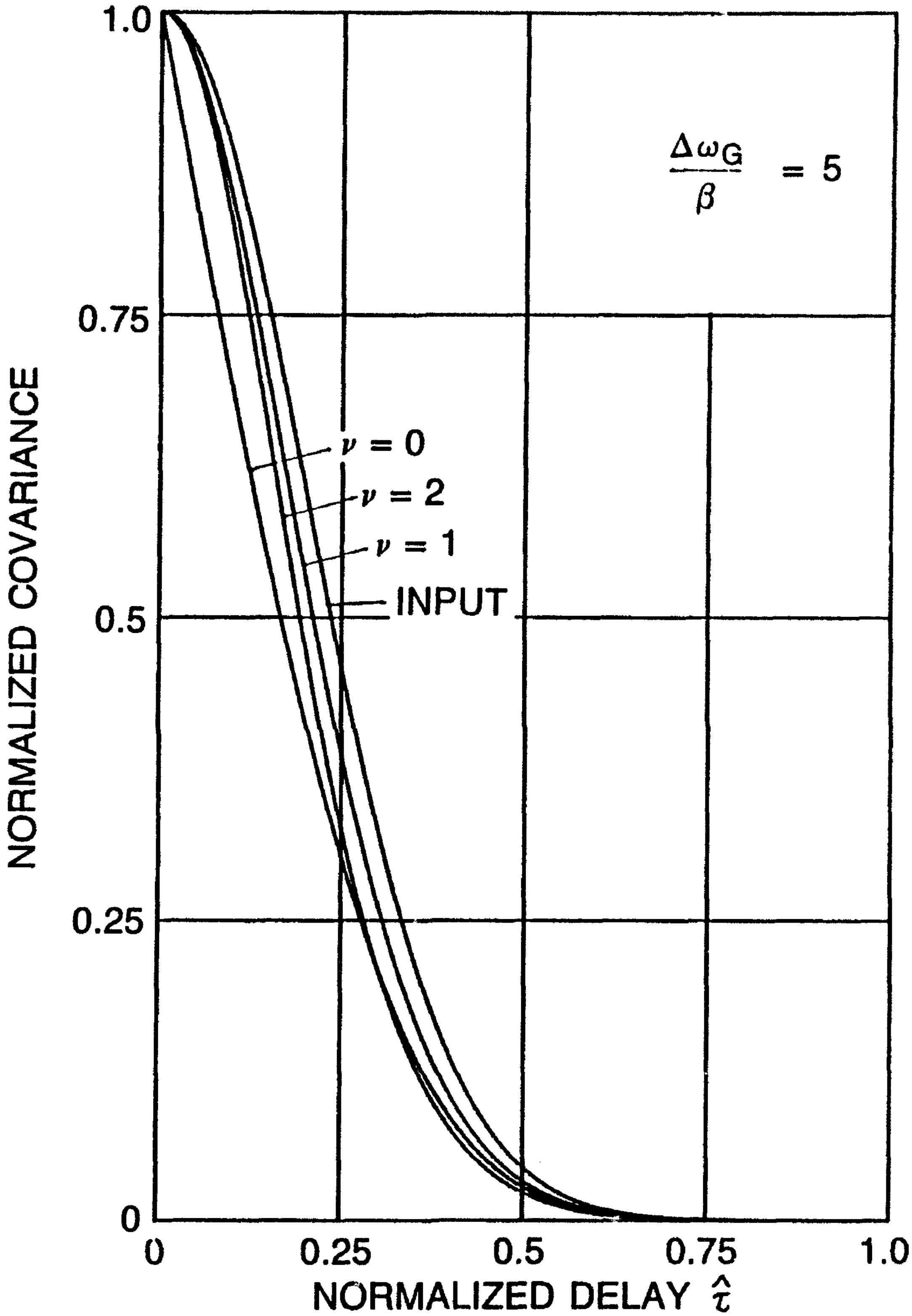

FIGURE 3.1 TEMPORAL COVARIANCE (FOR $\Delta\hat{R}=0$); GAUSS NOISE ONLY; CF. (2.11) WITH (2.7a), (2.9b), AND APPENDIX A.1

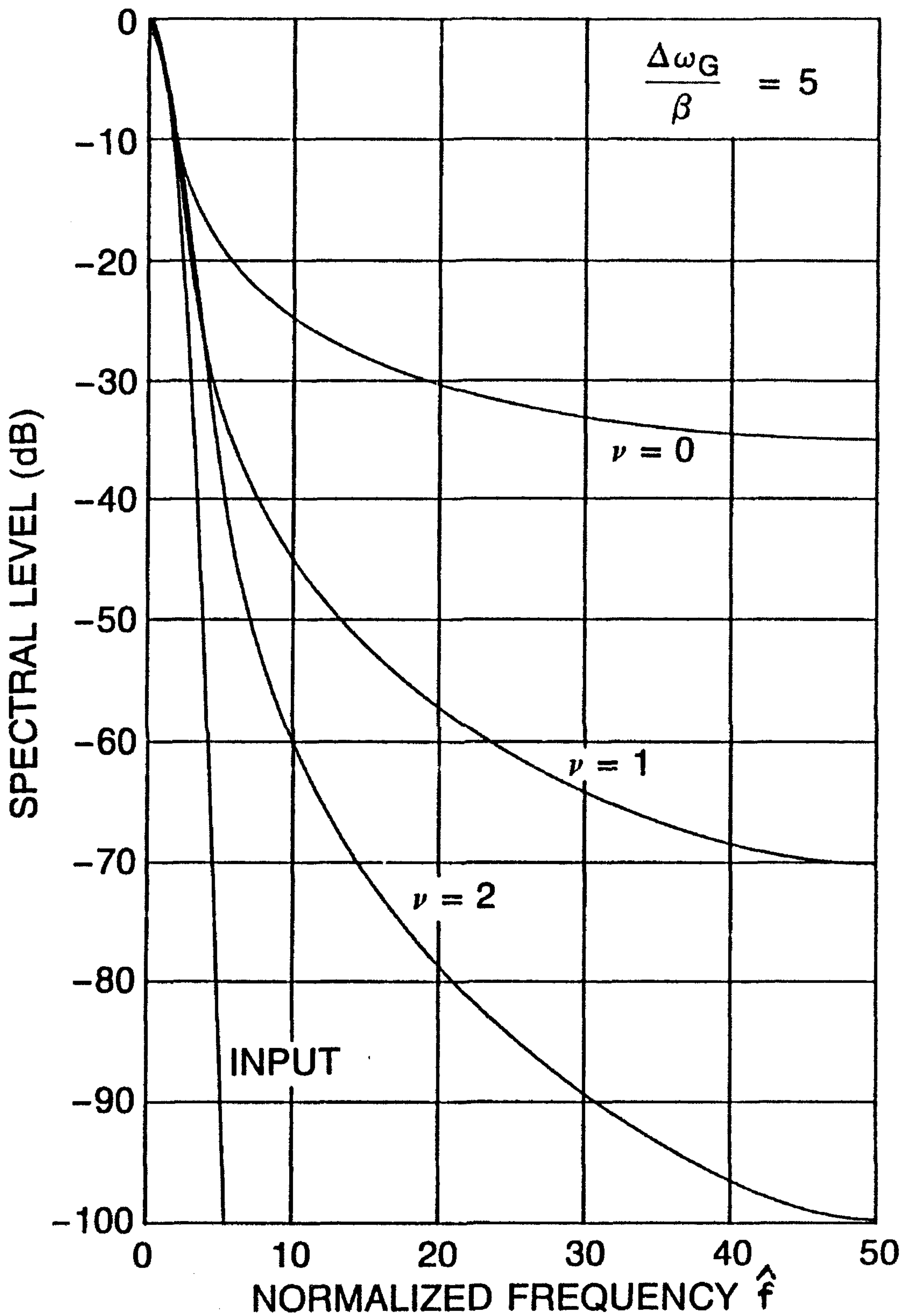

FIGURE 3.2 FREQUENCY (INTENSITY) SPECTRUM (FOR $\widehat{\Delta R}=0$); GAUSS NOISE ONLY; CF. (2.39), USED WITH (2.11), AND APPENDIX A.3

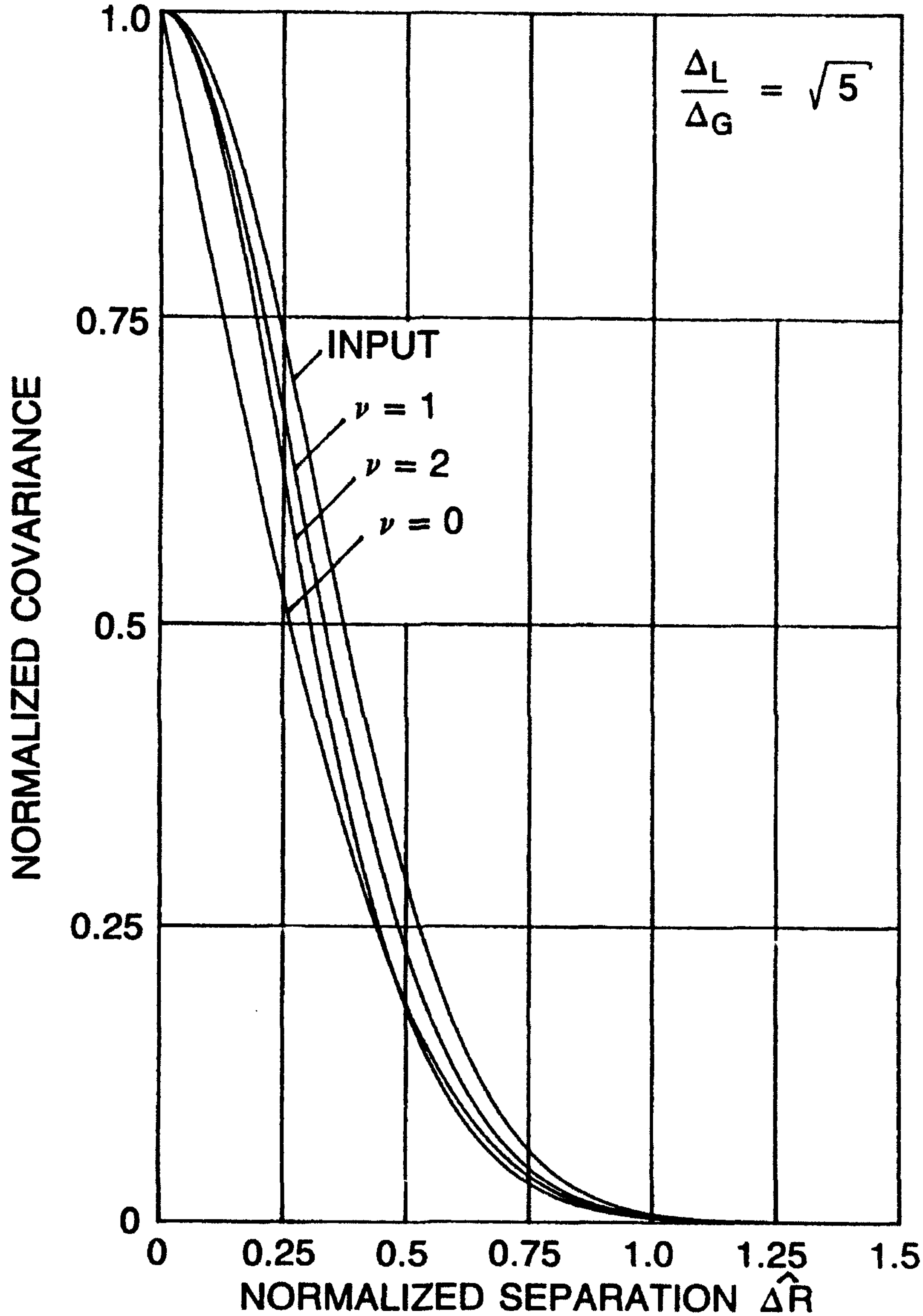

FIGURE 3.3 SPATIAL COVARIANCE (FOR $\hat{t}',\hat{t}=0$); GAUSS NOISE ONLY; CF. (2.11), (2.7b), AND APPENDIX A.2

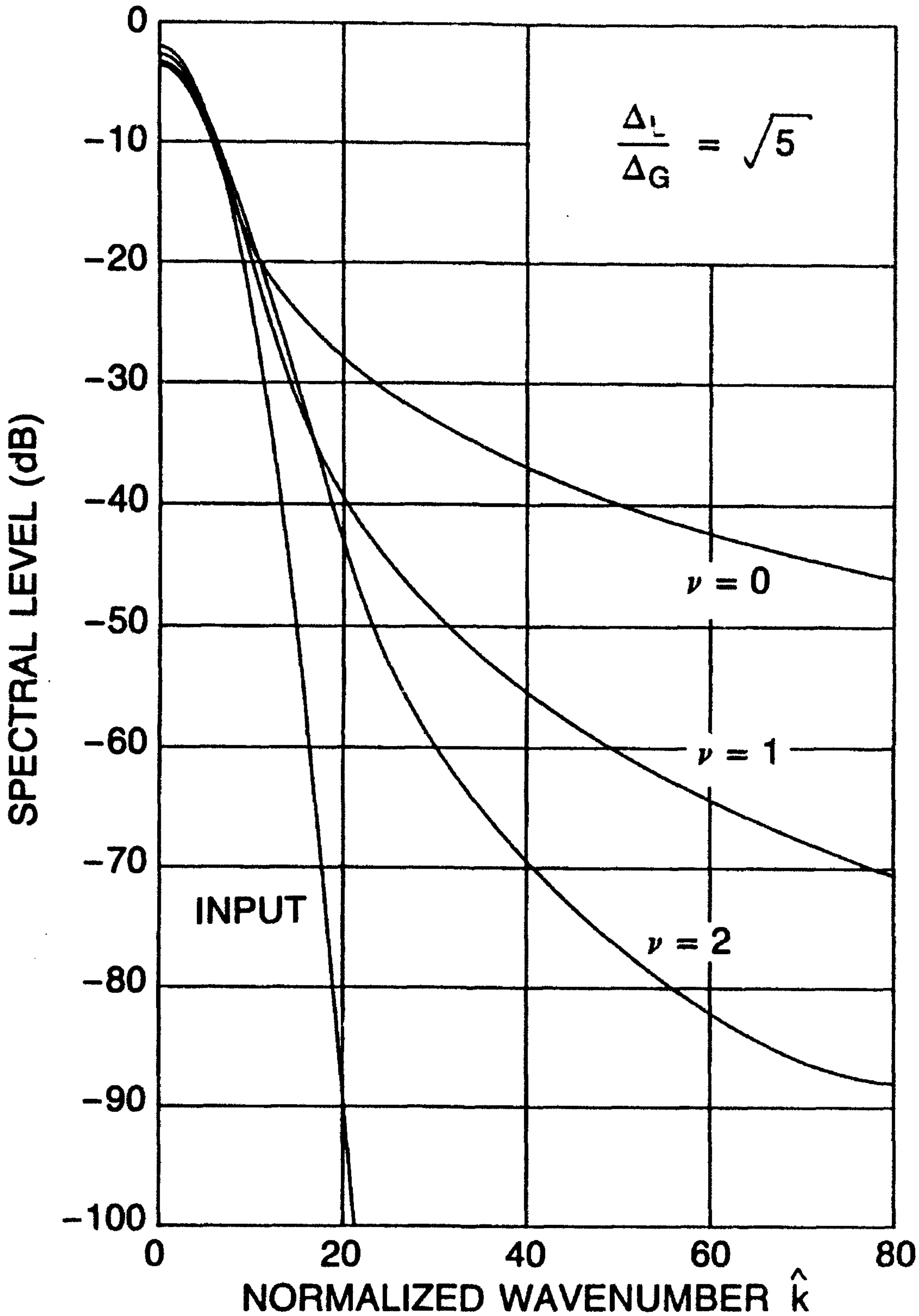

FIGURE 3.4 WAVENUMBER (INTENSITY) SPECTRUM (FOR $\hat{\tau}', \hat{\tau}=0$); GAUSS NOISE ONLY; CF. (2.35a,b) WITH (2.11), (2.7b), AND APPENDIX A.4

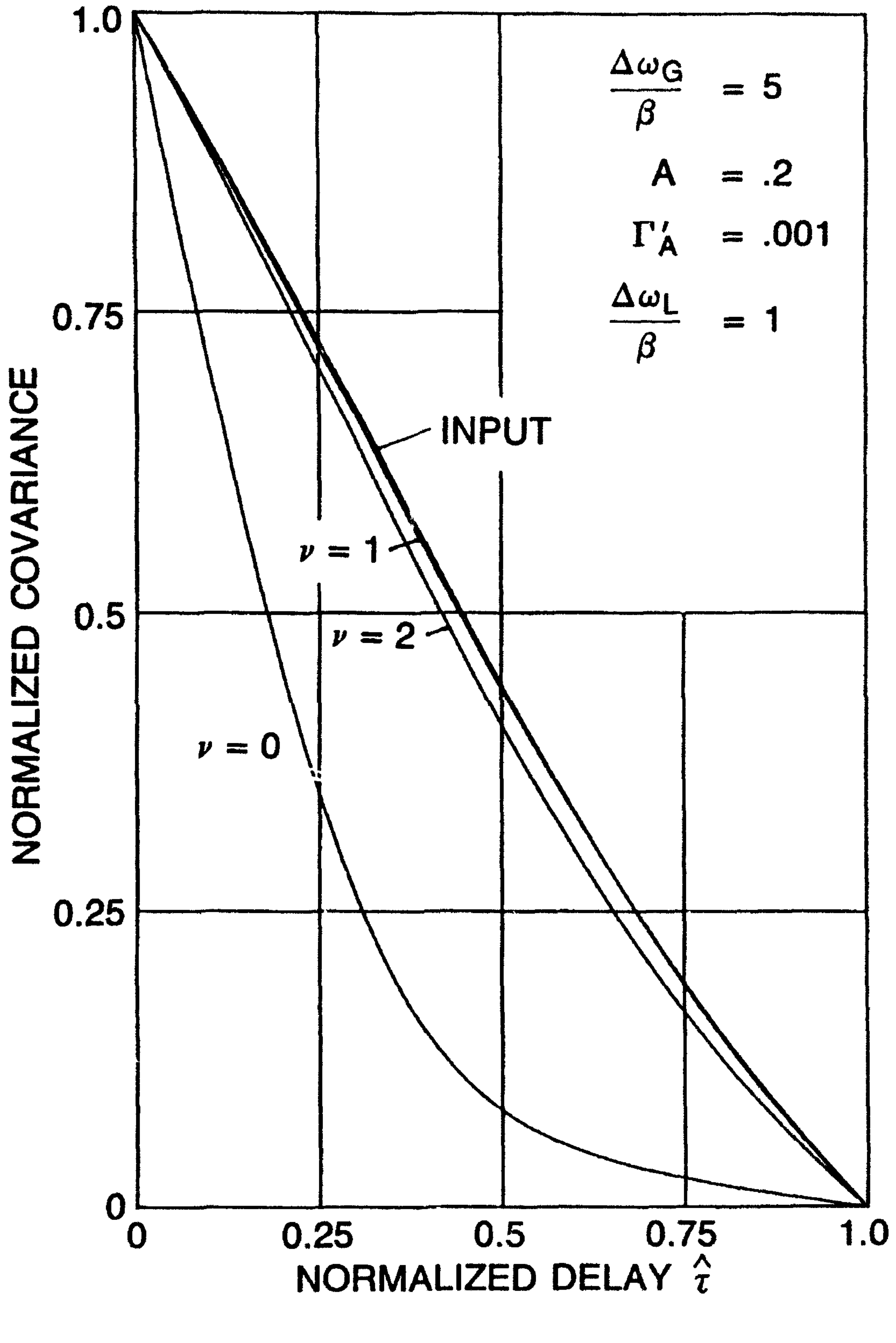

FIGURE 3.5 TEMPORAL COVARIANCE (FOR $\hat{\Delta R}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.9) AND APPENDIX A.1

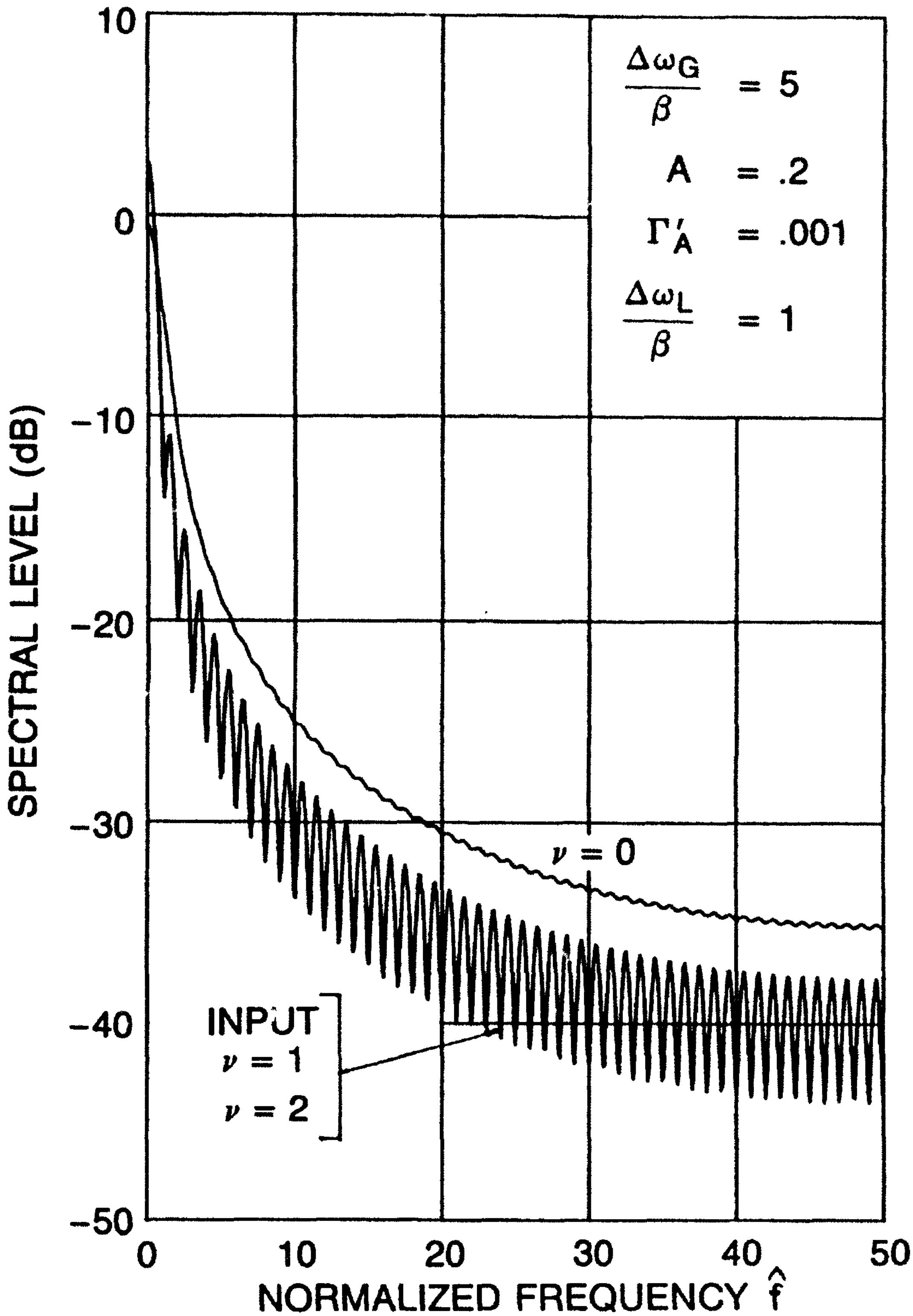

FIGURE 3.6 FREQUENCY (INTENSITY) SPECTRUM (FOR $\hat{\Delta R}=0$); CLASS A AND GAUSS NOISE; CF. (2.7) IN (2.39) WITH APPENDIX A.3

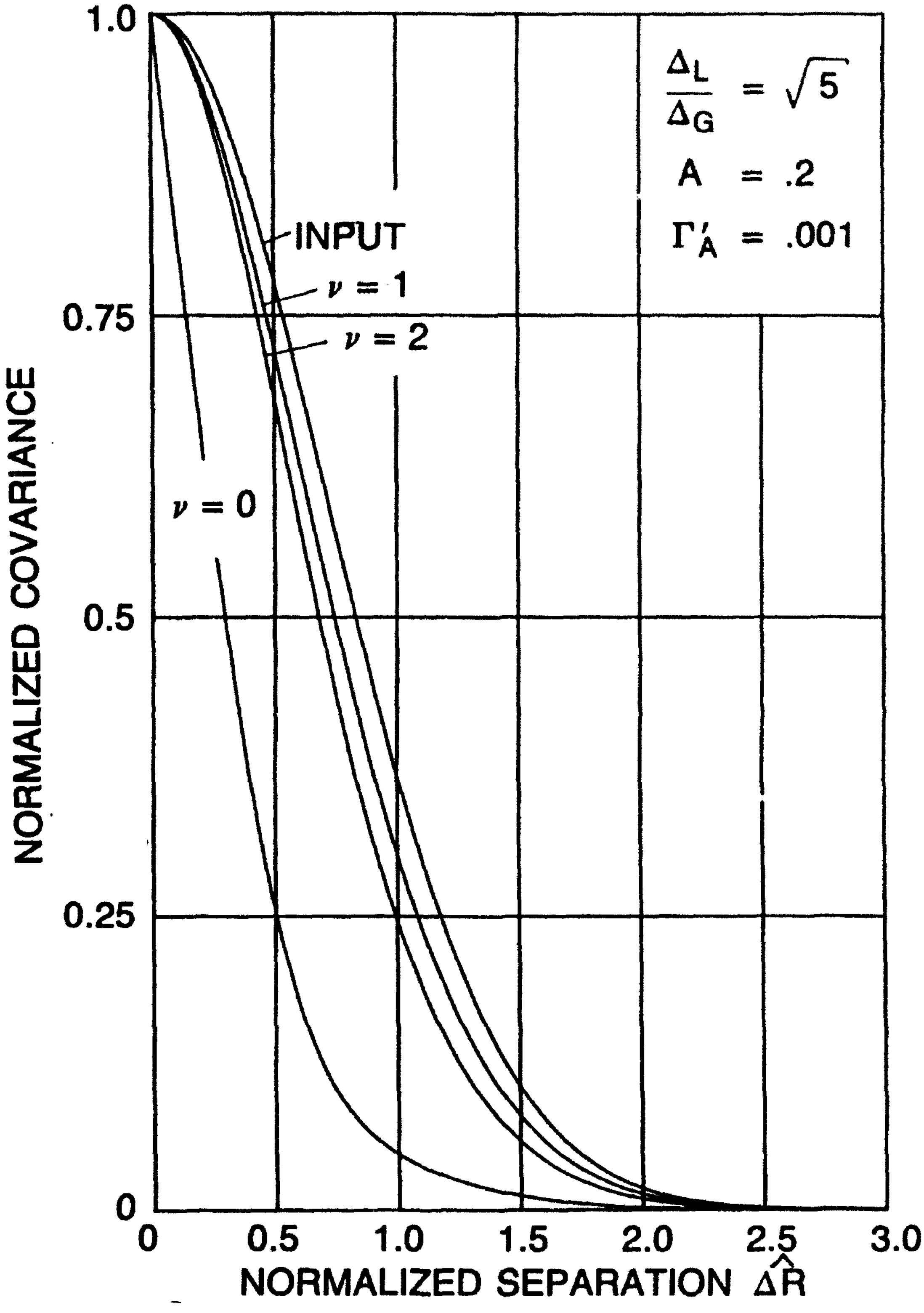

FIGURE 3.7 SPATIAL COVARIANCE (FOR $\hat{t}', \hat{t}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.10) WITH APPENDIX A.2

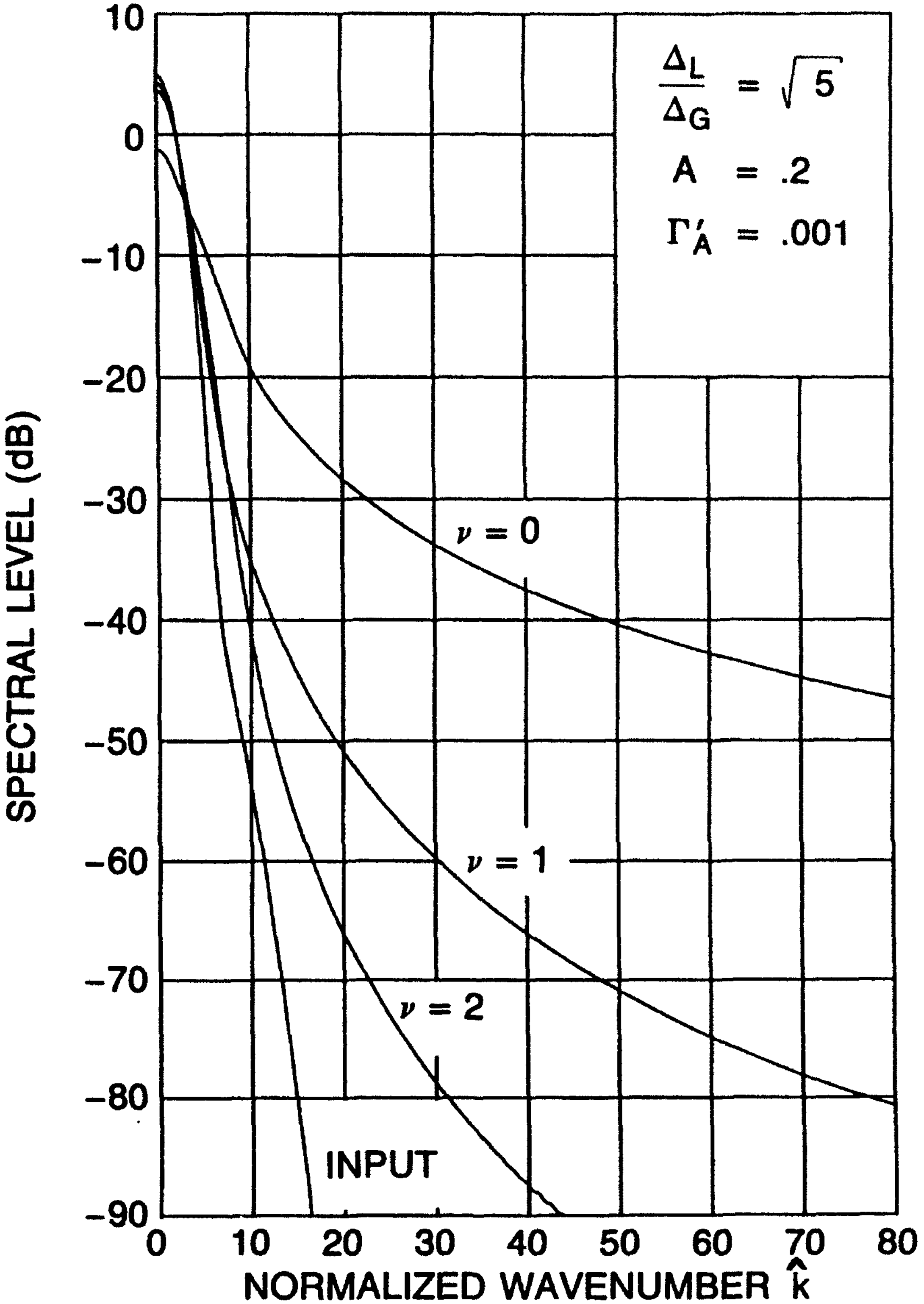

FIGURE 3.8 WAVENUMBER SPECTRUM (FOR $\hat{f}',\hat{t}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.10) IN (2.38a,b) AND APPENDIX A.4

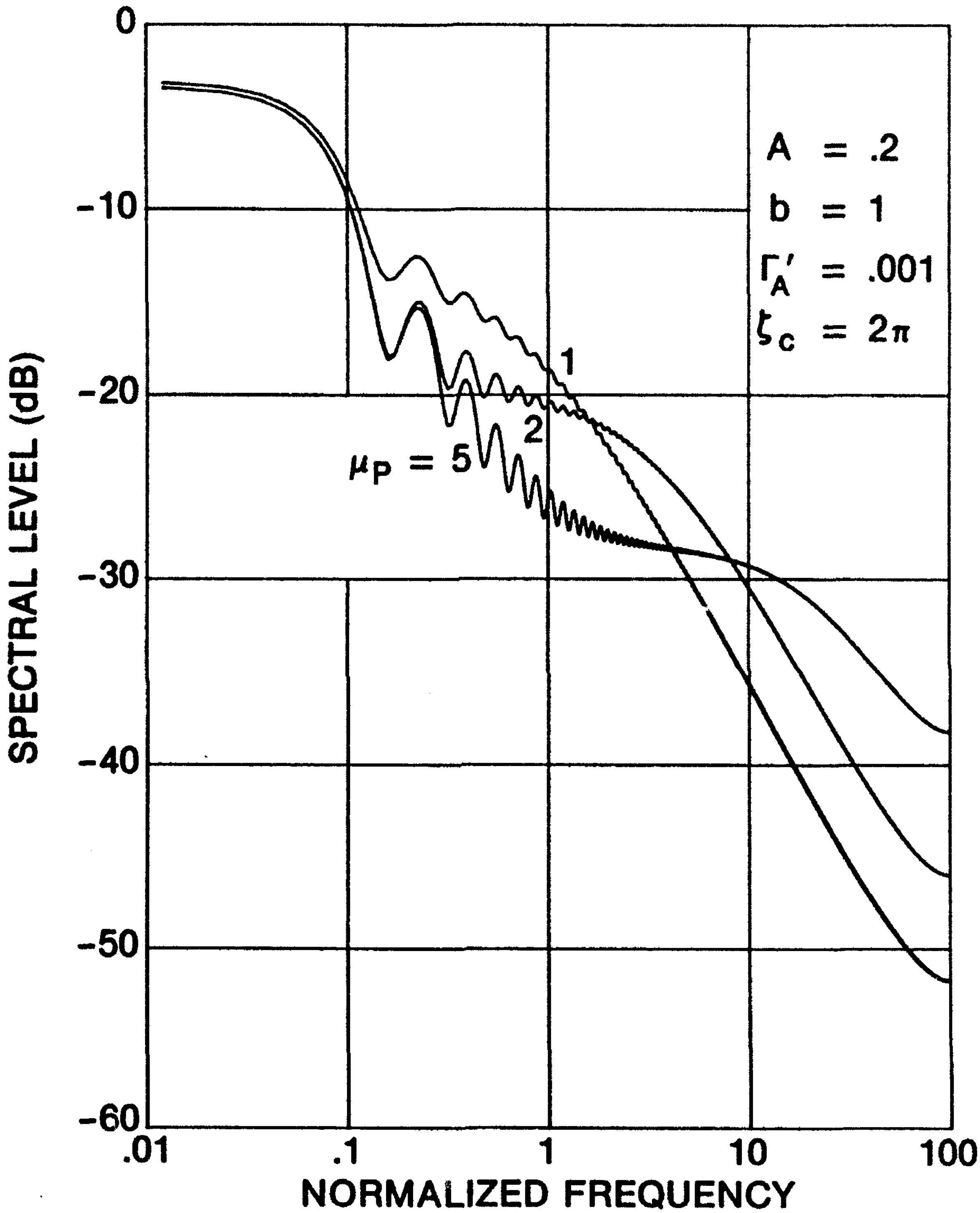

FIGURE 3.9a PHASE MODULATION (INTENSITY) SPECTRUM FOR INDEX μ_P=1,2,5, CLASS A AND GAUSS NOISE; CF. (2.50) WITH (2.44b), (2.45), (2.46) IN (2.52), AND APPENDIX A.5

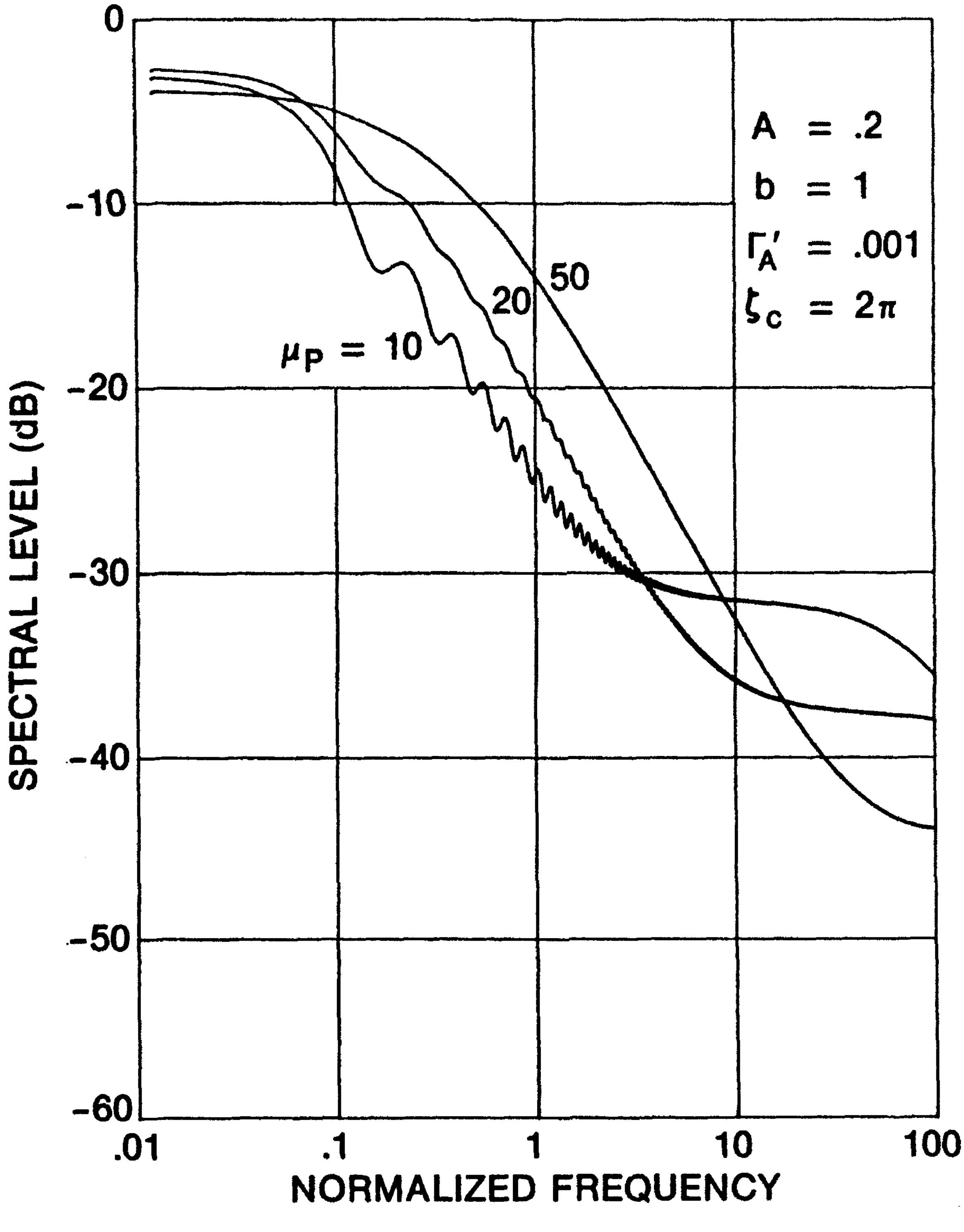

FIGURE 3.9b PHASE MODULATION (INTENSITY) SPECTRUM FOR INDEX μ_p=10,20,50, CLASS A AND GAUSS NOISE; CF. (2.50) WITH (2.44b), (2.45), (2.46) IN (2.52), AND APPENDIX A.5

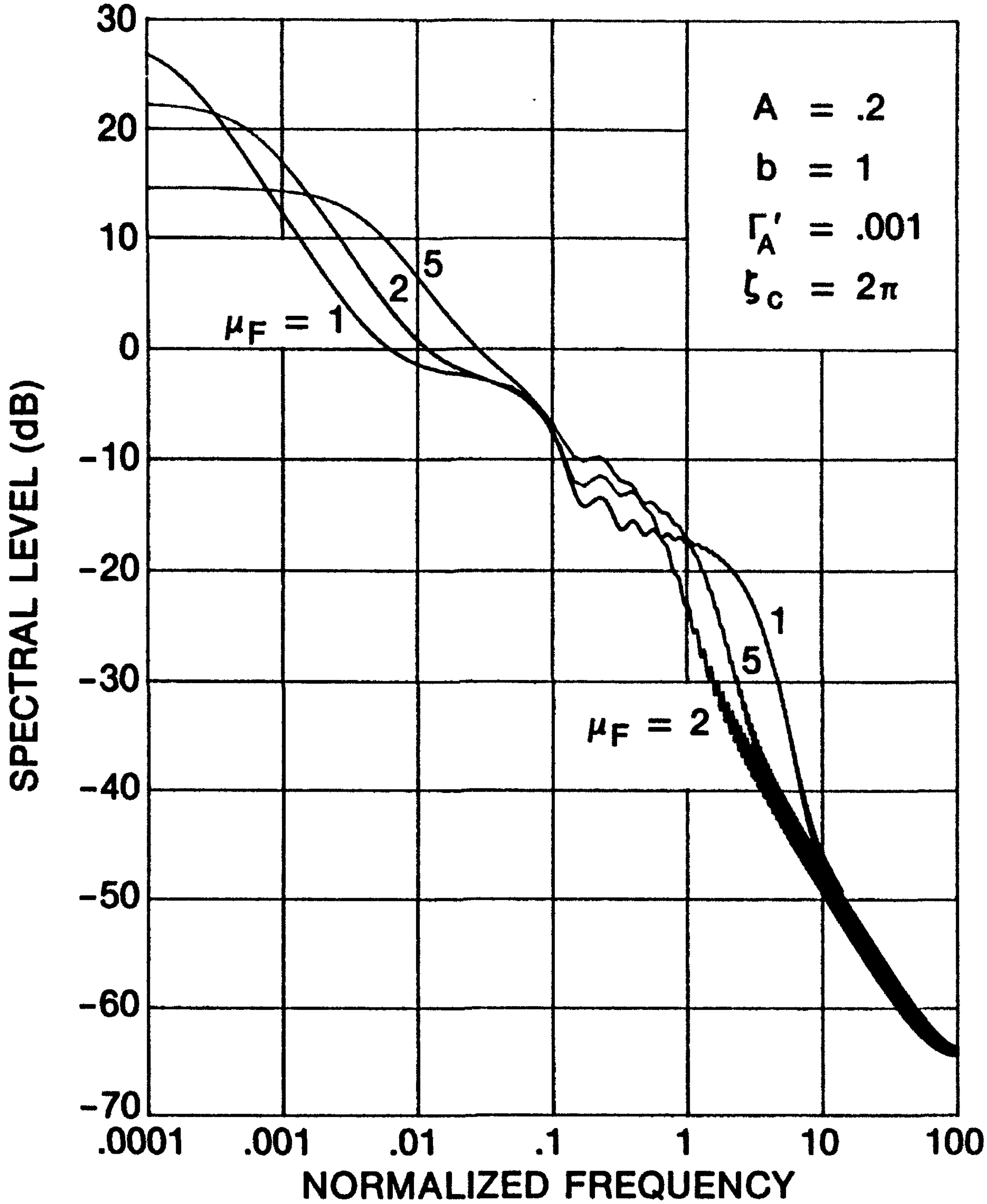

FIGURE 3.10a FREQUENCY MODULATION (INTENSITY) SPECTRUM FOR INDEX μ_F=1,2,5, CLASS A AND GAUSS NOISE; CF. (2.48) WITH (2.49), (2.44a), AND APPENDIX A.6

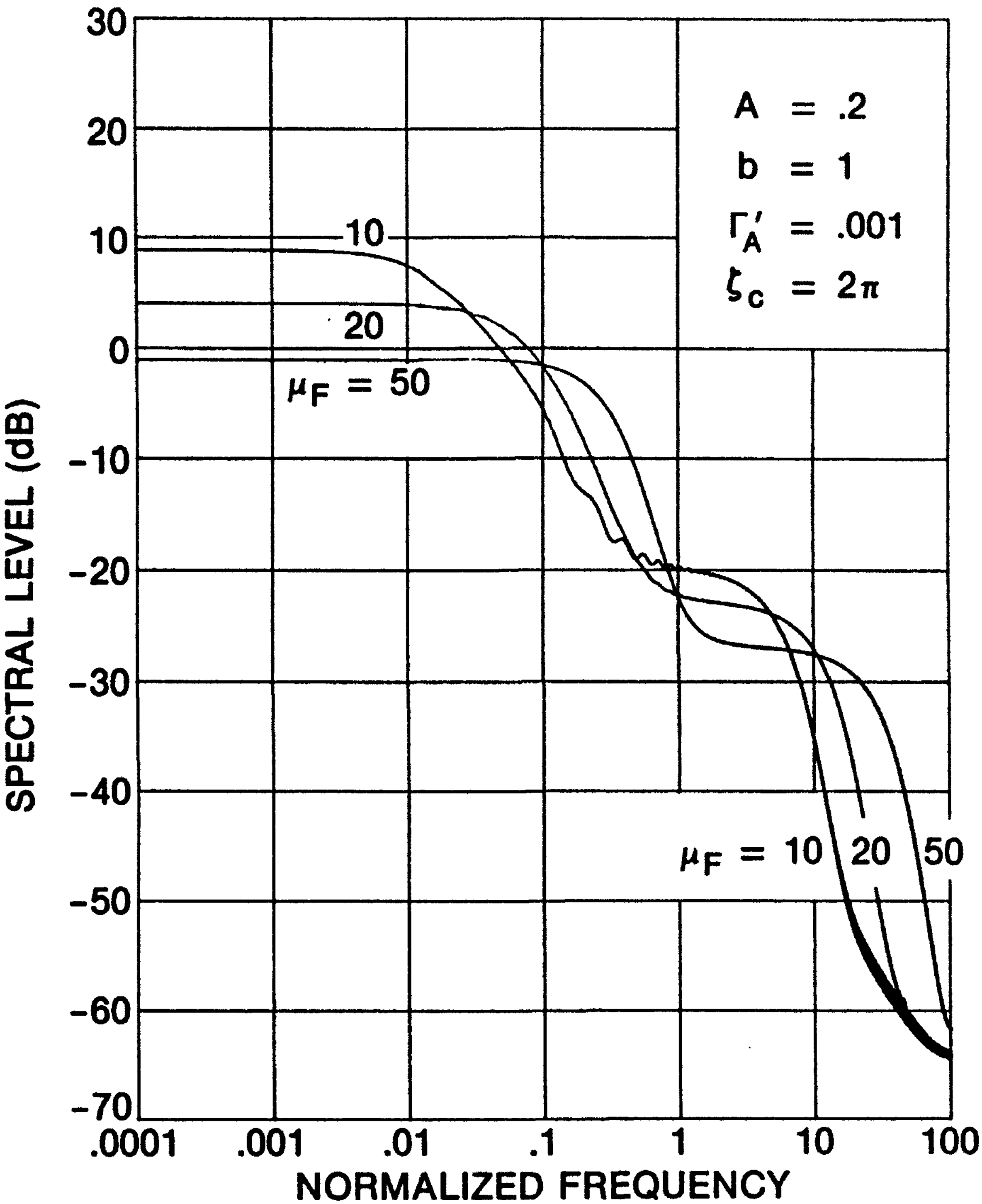

FIGURE 3.10b FREQUENCY MODULATION (INTENSITY) SPECTRUM FOR INDEX μ_F=10,20,50, CLASS A AND GAUSS NOISE; CF. (2.48) WITH (2.49), (2.44a), AND APPENDIX A.6

PART II. MATHEMATICAL AND COMPUTATIONAL PROCEDURES

4. SOME PROPERTIES OF THE COVARIANCE FUNCTION

In this section, we collect some useful relations for the covariance and auxiliary functions encountered in the numerical evaluation. These are necessary for rapid computation of the multiple series involved here and also serve as checks on the numerical procedures employed.

4.1 SIMPLIFICATION AND EVALUATION OF $B_\nu(Y)$

The function $B_\nu(Y)$ is defined by the following combination of hypergeometric functions:

$$B_\nu(Y) = \Gamma^2\left(\frac{\nu + 1}{2}\right) F\left(-\frac{\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}; Y^2\right) +$$

$$+ 2\,\Gamma^2\left(\frac{\nu}{2} + 1\right) Y\, F\left(\frac{1 - \nu}{2}, \frac{1 - \nu}{2}; \frac{3}{2}; Y^2\right) \quad \text{for} \quad Y^2 \le 1 \,. \qquad (4.1)$$

For the upper F function in (4.1), we have [1; (A.1.39b)]

$$\nu = 0, \quad F\left(0, 0; \frac{1}{2}; Y^2\right) = 1 \,;$$

$$\nu = 1, \quad F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; Y^2\right) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} \,;$$

$$\nu = 2, \quad F\left(-1, -1; \frac{1}{2}; Y^2\right) = 1 + 2Y^2 \,; \qquad (4.2)$$

where arcsin is the principal value inverse sine function. On the other hand, for the latter F function in (4.1), we have [1; (A.1.39a) and (A.1.39c)]

$$\nu = 0, \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; Y^2\right) = \frac{\arcsin(Y)}{Y} ;$$

$$\nu = 1, \quad F\left(0, 0; \frac{3}{2}; Y^2\right) = 1 ;$$

$$\nu = 2, \; F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; Y^2\right) = \frac{3}{4}\left(1 - Y^2\right)^{\frac{1}{2}} + \frac{1 + 2Y^2}{4Y} \arcsin(Y). \quad (4.3)$$

When these quantities are substituted in the above expression for $B_\nu(Y)$, we find the following relatively simple relations:

$$B_0(Y) = \pi + 2 \arcsin(Y) ,$$

$$B_1(Y) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} + \frac{\pi}{2}Y ,$$

$$B_2(Y) = \left(\frac{1}{2} + Y^2\right)\left(\frac{\pi}{2} + \arcsin(Y)\right) + \frac{3}{2}Y\left(1 - Y^2\right)^{\frac{1}{2}} . \quad (4.4)$$

These three quantities can be computed simultaneously by the following very compact computer coding in BASIC:

```
Y2=Y*Y
Sq=SQR(1.-Y2)
T=ASN(Y)+1.5707963267948966
B0=T+T
B1=Y*T+Sq
B2=(.5+Y2)*T+1.5*Y*Sq                                    (4.5)
```

Thus, the rather formidable expression, above, for $B_\nu(Y)$ can be evaluated by the use of just one square root and one arcsin when $\nu = 0, 1, 2$.

The following limiting values, which are obvious, are needed for various special cases:

$$\begin{aligned} B_0(0) &= \pi \ , & B_0(1) &= 2\pi \ , \\ B_1(0) &= 1 \ , & B_1(1) &= \pi \ , \\ B_2(0) &= \pi/4 \ , & B_2(1) &= 3\pi/2 \ , \\ B_3(0) &= 1 \ , & B_3(1) &= 15\pi/4 \ , \\ B_4(0) &= 9\pi/16 \ , & B_4(1) &= 105\pi/8 \ . \end{aligned} \tag{4.6}$$

These are special cases of

$$B_\nu(0) = \Gamma^2\left(\frac{\nu + 1}{2}\right) \ , \tag{4.7}$$

$$B_\nu(1) = 2\ \pi^{\frac{1}{2}}\ \Gamma\left(\nu + \frac{1}{2}\right) \ , \tag{4.8}$$

the latter following from [10; (15.1.20)].

4.2 LIMITING VALUES OF THE COVARIANCE FUNCTION

The covariance function at normalized separation $\widehat{\Delta R}$ and delay $\hat{\tau}$ is given by (2.7) as

$$\hat{M}_y(\widehat{\Delta R},\hat{\tau}) = \exp[-A(2-\rho)] \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{[A(1 - \rho)]^{m_1+m_2}}{m_1!\ m_2!}$$

$$\times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \left(\frac{n + m_1}{A} + \Gamma'_A\right)^{\nu/2} \left(\frac{n + m_2}{A} + \Gamma'_A\right)^{\nu/2} B_\nu(Y) \ , \tag{4.9}$$

where

$$\rho = \rho(\hat{\tau}) = \max\{0,\ 1 - |\hat{\tau}|\} \ , \tag{4.10}$$

$$Y = Y(m_1,m_2,n) = \frac{\frac{n}{A}\ k_L + \Gamma'_A\ k_G}{\left(\frac{n + m_1}{A} + \Gamma'_A\right)^{\frac{1}{2}} \left(\frac{n + m_2}{A} + \Gamma'_A\right)^{\frac{1}{2}}} \ , \tag{4.11}$$

$$k_L = k_L(\widehat{\Delta R},\hat{\tau}) = \exp\left[-\ \widehat{\Delta R}^2 - \frac{1}{2}\left(\frac{\Delta\omega_L}{\beta}\right)^2 \hat{\tau}^2\right] , \qquad (4.12)$$

$$k_G = k_G(\widehat{\Delta R},\hat{\tau}) = \exp\left[-\ \left(\frac{\Delta_L}{\Delta_G}\right)^2 \widehat{\Delta R}^2 - \frac{1}{2}\left(\frac{\Delta\omega_G}{\beta}\right)^2 \hat{\tau}^2\right] . \qquad (4.13)$$

The functions ρ, k_L, k_G can be replaced by other functional dependencies, if desired. The function $B_\nu(Y)$ has been considered earlier and considerably simplified for $\nu = 0, 1, 2$.

4.3 VALUE AT INFINITY

As $\widehat{\Delta R}$ or $\hat{\tau} \to \pm\infty$, then

$$\rho \to 0, \quad k_L \to 0, \quad k_G \to 0, \quad Y \to 0 . \qquad (4.14)$$

(If $|\hat{\tau}|$ remains less than 1 as $\widehat{\Delta R}$ tends to infinity, then ρ does not approach zero; this nuance has been discussed elsewhere in this report.) Then, it follows that

$$\hat{M}_y \to \exp(-2A) \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{A^{m_1+m_2}}{m_1!\ m_2!} \left(\frac{m_1}{A} + \Gamma'_A\right)^{\nu/2} \left(\frac{m_2}{A} + \Gamma'_A\right)^{\nu/2} B_\nu(0)$$

$$= B_\nu(0) \left(\exp(-A) \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\frac{m}{A} + \Gamma'_A\right)^{\nu/2}\right)^2 , \qquad (4.15)$$

because the sum on n can be terminated with the n = 0 term.

The sum on m can be effected in closed form, for $\nu = 0, 2, 4$, etc., by using the following results:

$$\sum_{m=0}^{\infty} \frac{A^m}{m!} = \exp(A) , \qquad (4.16)$$

$$\sum_{m=0}^{\infty} \frac{A^m}{m!}\, m = \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} = A \exp(A) \;, \tag{4.17}$$

$$\sum_{m=0}^{\infty} \frac{A^m}{m!}\, m^2 = \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!}\,(m - 1 + 1)$$

$$= \sum_{m=2}^{\infty} \frac{A^m}{(m-2)!} + \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} = (A^2 + A) \exp(A) \;. \tag{4.18}$$

There follows

$$\hat{M}_y(\infty) = \left\{ \begin{array}{ll} \pi & \text{for } \nu = 0 \\ \frac{\pi}{4}\left(1 + \Gamma'_A\right)^2 & \text{for } \nu = 2 \\ \frac{9\pi}{16}\left[\frac{1}{A} + \left(1 + \Gamma'_A\right)^2\right]^2 & \text{for } \nu = 4 \end{array} \right\} \;. \tag{4.19}$$

The case for $\nu = 1$ requires a numerical summation, once A and Γ'_A are specified. When these limiting values are subtracted from the correlation function, we obtain the covariance function.

4.4 VALUE AT THE ORIGIN

For $\widehat{\Delta R} = 0$, $\hat{t} = 0$, then

$$\rho = 1, \quad k_L = 1, \quad k_G = 1 \ , \tag{4.20}$$

and

$$\hat{M}_y(0,0) = \exp(-A) \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma_A'\right)^{\nu} B_\nu(1) \ , \tag{4.21}$$

because the sums on m_1 and m_2 can be terminated with the zero terms, thereby also leading to $Y = 1$.

The sum on n can be accomplished in closed form, for $\nu = 0, 1, 2$, etc., by using results given earlier. There follows

$$\hat{M}_y(0,0) = \begin{Bmatrix} 2\pi & \text{for } \nu = 0 \\ \pi\left(1 + \Gamma_A'\right) & \text{for } \nu = 1 \\ \frac{3}{2}\pi\left[\frac{1}{A} + \left(1 + \Gamma_A'\right)^2\right] & \text{for } \nu = 2 \end{Bmatrix} . \tag{4.22}$$

PART III. APPENDICES AND PROGRAMS

APPENDIX A.1 – EVALUATION OF COVARIANCE FUNCTION FOR ZERO SEPARATION ($\widehat{\Delta R} = 0$)

A program for the numerical evaluation of covariance $\hat{M}_y(\widehat{\Delta R},\hat{\tau})$ for $\widehat{\Delta R} = 0$ is contained in this appendix. Inputs required of the user are A, Γ_A', $(\Delta\omega_L/\beta)^2$, $(\Delta\omega_G/\beta)^2$, $\delta(\hat{\tau})$, $N(\hat{\tau})$, in lines 20 - 70. Since we are generally interested in values of A less than 1, the series for $\hat{M}_y$ in (4.9) will not have to be taken to very large values of m_1, m_2, n; accordingly, the values of $\{A^k/k!\}$ are tabulated once in lines 260 - 300 with a tolerance of 1E-10 set in line 80.

The values of the covariance at infinity, as given by (4.19), are computed and subtracted in lines 220 - 240 and 400 - 420; this is in anticipation of taking a Fourier transform of a covariance function which decays to zero for large arguments $\widehat{\Delta R}$.

The functions $B_\nu(Y)$ and $\hat{M}_y(\widehat{\Delta R},\hat{\tau})$ are available in the two subroutines starting at lines 1010 and 1120, respectively. The latter subroutine actually calculates the covariance at general nonzero values of both $\widehat{\Delta R}$ and $\hat{\tau}$, although we only employ it for $\widehat{\Delta R} = 0$ in this appendix; see lines 10 and 380. Also, for $\widehat{\Delta R} = 0$, the parameter Lg2 = $(\Delta_L/\Delta_G)^2$ is not relevant and, hence, is entered as zero in line 380.

The exponential Gaussian forms for k_L and k_G are used in lines 1200 and 1210, while the triangular form for ρ is entered in line 1240. Any of these can be replaced, if desired, by forms more appropriate to the user.

The program is written in BASIC for the Hewlett Packard 9000 Computer Model 520. The designation DOUBLE denotes integer variables, not double precision. The output from the program is stored in data files AOT0, AOT1, AOT2, for $\nu = 0, 1, 2$, respectively.

```
10    Rc=0.                             !  DelR^
20    A=.2                              !  A(subA)
30    Gp=.001                           !  GAMMA'(subA)
40    Wlb2=1.                           !  (DelW(subL)/Beta)^2
50    Wgb2=25.                          !  (DelW(subG)/Beta)^2
60    Dtc=.01                           !  INCREMENT IN Tau^
70    Ntc=200                           !  NUMBER OF Tau^ VALUES
80    Tolerance=1.E-10
90    COM Af(0:40),C(0:80),Sq(0:80)
100   COM DOUBLE J                      !  INTEGER
110   DIM Kag(200),Tc(0:200),F0(0:200),F1(0:200),F2(0:200)
120   DOUBLE Ntc,K                      !  INTEGERS
130   FOR K=0 TO Ntc
140   Tc=K*Dtc                          !  Tau^
150   Rho=MAX(0.,1.-ABS(Tc))            !  Rho
160   T2=.5*Tc*Tc
170   Kl=EXP(-Wlb2*T2)
180   Kg=EXP(-Wgb2*T2)
190   Kag(K)=(Rho*Kl+Gp*Kg)/(1.+Gp)      !  INPUT COVARIANCE
200   NEXT K
210   A1=1./A                           !  A>0 REQUIRED
220   F0inf=PI
230   F1inf=FNF1inf(A,Gp)
240   F2inf=.25*PI*(1.+Gp)*(1.+Gp)
250   Af(0)=1.
260   FOR K=1 TO 40
270   J=K
280   Af(K)=T=Af(K-1)*A/K               !  A^K/K!
290   IF T<Tolerance THEN 320
300   NEXT K
310   PRINT "40 TERMS IN Af(*)"
320   FOR K=0 TO J*2
330   C(K)=T=K*A1+Gp
340   Sq(K)=1./SQR(T)
350   NEXT K
360   FOR K=0 TO Ntc
370   Tc(K)=Tc=K*Dtc                    !  Tau^
380   CALL Myc(Rc,Tc,A,Gp,Wlb2,Wgb2,0.,F0(K),F1(K),F2(K))
390   NEXT K
400   MAT F0=F0-(F0inf)
410   MAT F1=F1-(F1inf)
420   MAT F2=F2-(F2inf)
430   MAT F0=F0/(F0(0))
440   MAT F1=F1/(F1(0))
450   MAT F2=F2/(F2(0))
460   PRINT "INFINITY:";F0inf;F1inf;F2inf
470   PRINT "MINIMA:  ";MIN(F0(*));MIN(F1(*));MIN(F2(*))
480   PRINT "AT Ntc:  ";F0(Ntc);F1(Ntc);F2(Ntc)
490   CREATE DATA "AIT1",8
500   ASSIGN #1 TO "AIT1"
510   PRINT #1;Kag(*)
520   CREATE DATA "AOT0",8
530   ASSIGN #1 TO "AOT0"
540   PRINT #1;F0(*)
550   CREATE DATA "AOT1",8
```

```
560     ASSIGN #1 TO "AOT1"
570     PRINT #1;F1(*)
580     CREATE DATA "AOT2",8
590     ASSIGN #1 TO "AOT2"
600     PRINT #1;F2(*)
610     ASSIGN #1 TO *
620     Tcmax=Dtc*Ntc
630     GINIT 200/260
640     PLOTTER IS 505,"HPGL"
650     PRINTER IS 505
660     LIMIT PLOTTER 505,0,200,0,260
670     VIEWPORT 22,85,19,122
680     WINDOW 0.,1.,0.,1.
690        PRINT "VS5"
700     GRID .25,.25
710        PRINT "VS36"
720     PLOT Tc(*),Kag(*)
730     PENUP
740     PLOT Tc(*),F0(*)
750     PENUP
760     PLOT Tc(*),F1(*)
770     PENUP
780     PLOT Tc(*),F2(*)
790     PENUP
800     PAUSE
810     PRINTER IS CRT
820     PLOTTER 505 IS TERMINATED
830     END
840     !
850     DEF FNF1inf(A,Gp)            !  for v(=nu) = 1
860     Tol=1.E-18
870     Ag=A*Gp
880     T=1.
890     S=SQR(1.+Ag)
900     FOR M=2 TO 100
910     T=T*A/M
920     P=T*SQR(M+Ag)
930     S=S+P
940     IF P<S*Tol THEN 970
950     NEXT M
960     PRINT "100 TERMS IN FNF1inf"
970     T=Gp+A*S*S+2.*SQR(Ag)*S
980     RETURN EXP(-2.*A)*T
990     FNEND
1000    !
1010    SUB Bnu(Y,B0,B1,B2)          !  Bv(Y) for v=0,1,2
1020    IF Y>1. THEN PRINT "Y = 1 +";Y-1.
1030    IF Y>1. THEN Y=1.
1040    Y2=Y*Y
1050    Sq=SQR(1.-Y2)
1060    T=ASN(Y)+1.5707963267948966
1070    B0=T+T
1080    B1=Y*T+Sq
1090    B2=(.5+Y2)*T+1.5*Y*Sq
1100    SUBEND
1110    !
```

```
1120    SUB Myc(Rc,Tc,A,Gp,W1b2,Wgb2,Lg2,S0,S1,S2)
1130    COM Af(*),C(*),Sq(*)
1140    COM DOUBLE J                      !   INTEGER
1150    ALLOCATE Ap(0:J),Ap1(0:J)
1160    DOUBLE K,M1,M2,N,K1,K2            !   INTEGERS
1170    A1=1./A                           !   A>0 REQUIRED
1180    T2=.5*Tc*Tc
1190    R2=Rc*Rc
1200    K1=EXP(-R2-W1b2*T2)
1210    Kg=EXP(-Lg2*R2-Wgb2*T2)
1220    Ak=A1*K1
1230    Gk=Gp*Kg
1240    Rho=MAX(0.,1.-ABS(Tc))            !   Rho
1250    Rho1=1.-Rho
1260    Ap(0)=Ap1(0)=Pk=Pk1=1.
1270    FOR K=1 TO J
1280    Pk=Pk*Rho
1290    Pk1=Pk1*Rho1
1300    T=Af(K)
1310    Ap(K)=T*Pk
1320    Ap1(K)=T*Pk1
1330    NEXT K
1340    S0m1=S1m1=S2m1=0.
1350    FOR M1=0 TO J
1360    S0m2=S1m2=S2m2=0.
1370    FOR M2=0 TO J
1380    S0n=S1n=S2n=0.
1390    FOR N=0 TO J
1400    K1=N+M1
1410    K2=N+M2
1420    T=Ap(N)
1430    P=C(K1)*C(K2)
1440    Y=(N*Ak+Gk)*Sq(K1)*Sq(K2)
1450    CALL Bnu(Y,B0,B1,B2)
1460    S0n=S0n+T*B0
1470    S1n=S1n+T*SQR(P)*B1
1480    S2n=S2n+T*P*B2
1490    NEXT N
1500    T2=Ap1(M2)
1510    S0m2=S0m2+T2*S0n
1520    S1m2=S1m2+T2*S1n
1530    S2m2=S2m2+T2*S2n
1540    NEXT M2
1550    T1=Ap1(M1)
1560    S0m1=S0m1+T1*S0m2
1570    S1m1=S1m1+T1*S1m2
1580    S2m1=S2m1+T1*S2m2
1590    NEXT M1
1600    T=EXP(-A*(2.-Rho))
1610    S0=T*S0m1
1620    S1=T*S1m1
1630    S2=T*S2m1
1640    SUBEND
```

APPENDIX A.2 — EVALUATION OF COVARIANCE FUNCTION FOR ZERO DELAY ($\hat{\tau},\hat{\tau}' = 0$)

A program for the numerical evaluation of covariance $\hat{M}_y(\widehat{\Delta R},\hat{\tau})$ for $\hat{\tau},\hat{\tau}' = 0$ is contained in this appendix. Inputs required of the user are A, Γ_A', $(\Delta_L/\Delta_G)^2$, $\delta(\widehat{\Delta R})$, $N(\widehat{\Delta R})$, in lines 20 - 60. The tolerance for terminating the triple infinite sums is set at 1E-15 in line 70. The output from the program is stored in data files AOR0, AOR1, AOR2, for $\nu = 0, 1, 2$, respectively. Other relevant comments are made in appendix A.1.

The limit of $\hat{M}_y$ at $\widehat{\Delta R} = \infty$ (when $\hat{\tau} = 0$) is given by the closed form results

$$\hat{M}_y(\infty,0) = \begin{cases} \pi & \text{for } \nu = 0 \\ 1 + \Gamma_A' & \text{for } \nu = 1 \\ \frac{\pi}{4}\left[\frac{1}{A} + \left(1 + \Gamma_A'\right)^2\right] & \text{for } \nu = 2 \end{cases} . \qquad (A.2\text{-}1)$$

These values have been subtracted from $\hat{M}_y$ so that we can Bessel transform a function which tends to zero as $\widehat{\Delta R} \to \infty$.

```
10   Tc=0.                          !  Tau^
20   A=.2                           !  A(subA)
30   Gp=.001                        !  GAMMA'(subA)
40   Lg2=5.                         !  (DelL/DelG)^2
50   Drc=.005                       !  INCREMENT IN DelR^
60   Nrc=900                        !  NUMBER OF DelR^ VALUES
70   Tolerance=1.E-15
80   COM Af(0:40),C(0:80),Sq(0:80)
90   COM DOUBLE J                   !  INTEGER
100  DIM Rc(0:900),Kag(0:900),F0(0:900),F1(0:900),F2(0:900)
110  DOUBLE Nrc,K                   !  INTEGERS
120  A1=1./A                        !  A>0 REQUIRED
130  F0inf=PI                                 !  LIMITS FOR
140  F1inf=1.+Gp                              !  Rc=infinity
150  F2inf=.25*PI*((1.+Gp)*(1.+Gp)+A1)        !  AND Tc=0
160  Af(0)=1.
170  FOR K=1 TO 40
180  J=K
190  Af(K)=T=Af(K-1)*A/K            !  A^K/K!
200  IF T<Tolerance THEN 230
210  NEXT K
220  PRINT "40 TERMS IN Af(*)"
230  FOR K=0 TO J*2
240  C(K)=T=K*A1+Gp
250  Sq(K)=1./SQR(T)
260  NEXT K
```

```
 270    FOR K=0 TO Nrc
 280    Rc(K)=Rc=K*Drc                  !  DelR^
 290    R2=Rc*Rc
 300    Kl=EXP(-R2)
 310    Kg=EXP(-Lg2*R2)
 320    Rho=MAX(0.,1.-ABS(Tc))          !  Rho
 330    Kag(K)=(Rho*Kl+Gp*Kg)/(1.+Gp)   !  INPUT COVARIANCE
 340    CALL Myc(Rc,Tc,A,Gp,0.,0.,Lg2,F0(K),F1(K),F2(K))
 350    NEXT K
 360    MAT F0=F0-(F0inf)
 370    MAT F1=F1-(F1inf)
 380    MAT F2=F2-(F2inf)
 390    MAT F0=F0/(F0(0))
 400    MAT F1=F1/(F1(0))
 410    MAT F2=F2/(F2(0))
 420    PRINT "INFINITY:";F0inf;F1inf;F2inf
 430    PRINT "MINIMA:  ";MIN(F0(*));MIN(F1(*));MIN(F2(*))
 440    PRINT "AT Nrc:  ";F0(Nrc);F1(Nrc);F2(Nrc)
 450    CREATE DATA "AIR1",33
 460    ASSIGN #1 TO "AIR1"
 470    PRINT #1;Kag(*)
 480    CREATE DATA "AOR0",33
 490    ASSIGN #1 TO "AOR0"
 500    PRINT #1;F0(*)
 510    CREATE DATA "AOR1",33
 520    ASSIGN #1 TO "AOR1"
 530    PRINT #1;F1(*)
 540    CREATE DATA "AOR2"
 550    ASSIGN #1 TO "AOR2
 560    PRINT #1;F2(*)
 570    ASSIGN #1 TO *
 580    Rcmax=Drc*Nrc
 590    GINIT 200/260
 600    PLOTTER IS 505,"HPGL"
 610    PRINTER IS 505
 620    LIMIT PLOTTER 505,0,200,0,260
 630    VIEWPORT 22,85,19,122
 640    WINDOW 0.,3.,0.,1.
 650       PRINT "VS5"
 660    GRID .5,.25
 670       PRINT "VS36"
 680    PLOT Rc(*),Kag(*)
 690    PENUP
 700    PLOT Rc(*),F0(*)
 710    PENUP
 720    PLOT Rc(*),F1(*)
 730    PENUP
 740    PLOT Rc(*),F2(*)
 750    PENUP
 760    PAUSE
 770    PRINTER IS CRT
 780    PLOTTER 505 IS TERMINATED
 790    END
 800    !
 810    SUB Bnu(Y,B0,B1,B2)             !  Bv(Y) for v=0,1,2
 820    ! SEE APPENDIX A.1
 900    SUBEND
 910    !
 920    SUB Myc(Rc,Tc,A,Gp,Wlb2,Wgb2,Lg2,S0,S1,S2)
 930    ! SEE APPENDIX A.1
1440    SUBEND
```

APPENDIX A.3 – EVALUATION OF TEMPORAL INTENSITY SPECTRUM FOR ZERO SEPARATION ($\widehat{\Delta R} = 0$)

A program for the numerical evaluation of the Fourier transform of covariance $\hat{M}_y(0,\hat{\tau}) - \hat{M}_y(\infty)$ is contained in this appendix. Inputs required of the user are listed in lines 10 - 30. The data input, AOT0 or AOT1 or AOT2, as generated by means of the program in appendix A.1, is injected by means of lines 410, 600, and 790.

In order to keep the FFT (fast Fourier transform) size, N in lines 30 and 320, at reasonable values, the data sequence is collapsed, without any loss of accuracy, according to the method given in [8; pages 7 - 8] and [9; pages 13 - 16]. The integration rule documented here is the trapezoidal rule; this procedure is very accurate and efficient and is recommended for numerical Fourier transforms.

```
 10    Ntc=200                    !  NUMBER OF Tau^ VALUES
 20    Dtc=.01                    !  INCREMENT IN Tau^
 30    N=1024                     !  SIZE OF FFT; N > Ntc REQUIRED
 40    DOUBLE Ntc,N,N4,N2,Ns      !  INTEGERS
 50    N4=N/4
 60    N2=N/2
 70    REDIM Cos(0:N4),X(0:N-1),Y(0:N-1)
 80    DIM Cos(256),X(1023),Y(1023),A(200)
 90    T=2.*PI/N
100    FOR Ns=0 TO N4
110    Cos(Ns)=COS(T*Ns)          !  QUARTER-COSINE TABLE IN Cos(*)
120    NEXT Ns
130    GINIT 200/260
140    PLOTTER IS 505,"HPGL"
150    PRINTER IS 505
160    LIMIT PLOTTER 505,0,200,0,260
170    VIEWPORT 22,85,19,122
180    WINDOW 0,N2,-5,1
190       PRINT "VS5"
200    GRID N/10,1
210       PRINT "VS36"
220    ASSIGN #1 TO "AIT1"
230    READ #1;A(*)
240    MAT X=(0.)
250    MAT Y=(0.)
260    X(0)=.5*A(0)
270    FOR Ns=1 TO Ntc-1
280    X(Ns)=A(Ns)
290    NEXT Ns
300    X(Ntc)=.5*A(Ntc)
```

```
310     MAT X=X*(Dtc*4.)
320     CALL Fft14(N,Cos(*),X(*),Y(*))
330     FOR Ns=0 TO N2
340     Ar=X(Ns)
350     IF Ar>0. THEN 380
360     PENUP
370     GOTO 390
380     PLOT Ns,LGT(Ar)
390     NEXT Ns
400     PENUP
410     ASSIGN #1 TO "AOT0"
420     READ #1;A(*)
430     MAT X=(0.)
440     MAT Y=(0.)
450     X(0)=.5*A(0)
460     FOR Ns=1 TO Ntc-1
470     X(Ns)=A(Ns)
480     NEXT Ns
490     X(Ntc)=.5*A(Ntc)
500     MAT X=X*(Dtc*4.)
510     CALL Fft14(N,Cos(*),X(*),Y(*))
520     FOR Ns=0 TO N2
530     Ar=X(Ns)
540     IF Ar>0. THEN 570
550     PENUP
560     GOTO 580
570     PLOT Ns,LGT(Ar)
580     NEXT Ns
590     PENUP
600     ASSIGN #1 TO "AOT1"
610     READ #1;A(*)
620     MAT X=(0.)
630     MAT Y=(0.)
640     X(0)=.5*A(0)
650     FOR Ns=1 TO Ntc-1
660     X(Ns)=A(Ns)
670     NEXT Ns
680     X(Ntc)=.5*A(Ntc)
690     MAT X=X*(Dtc*4.)
700     CALL Fft14(N,Cos(*),X(*),Y(*))
710     FOR Ns=0 TO N2
720     Ar=X(Ns)
730     IF Ar>0. THEN 760
740     PENUP
750     GOTO 770
760     PLOT Ns,LGT(Ar)
770     NEXT Ns
780     PENUP
790     ASSIGN #1 TO "AOT2"
800     READ #1;A(*)
810     MAT A=A/(A(0))
820     MAT X=(0.)
830     MAT Y=(0.)
```

```
 840   X(0)=.5*A(0)
 850   FOR Ns=1 TO Ntc-1
 860   X(Ns)=A(Ns)
 870   NEXT Ns
 880   X(Ntc)=.5*A(Ntc)
 890   MAT X=X*(Dtc*4.)
 900   CALL Fft14(N,Cos(*),X(*),Y(*))
 910   FOR Ns=0 TO N2
 920   Ar=X(Ns)
 930   IF Ar>0. THEN 960
 940   PENUP
 950   GOTO 970
 960   PLOT Ns,LGT(Ar)
 970   NEXT Ns
 980   PENUP
 990   PAUSE
1000   END
1010   !
1020   SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1030   DOUBLE Log2n,N1,N2,N3,N4,J,K   !  INTEGERS < 2^31 = 2,147,483,648
1040   DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1050   IF N=1 THEN SUBEXIT
1060   IF N>2 THEN 1140
1070   A=X(0)+X(1)
1080   X(1)=X(0)-X(1)
1090   X(0)=A
1100   A=Y(0)+Y(1)
1110   Y(1)=Y(0)-Y(1)
1120   Y(0)=A
1130   SUBEXIT
1140   A=LOG(N)/LOG(2.)
1150   Log2n=A
1160   IF ABS(A-Log2n)<1.E-8 THEN 1190
1170   PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
1180   PAUSE
1190   N1=N/4
1200   N2=N1+1
1210   N3=N2+1
1220   N4=N3+N1
1230   FOR I1=1 TO Log2n
1240   I2=2^(Log2n-I1)
1250   I3=2*I2
1260   I4=N/I3
1270   FOR I5=1 TO I2
1280   I6=(I5-1)*I4+1
1290   IF I6<=N2 THEN 1330
1300   A1=-Cos(N4-I6-1)
1310   A2=-Cos(I6-N1-1)
1320   GOTO 1350
1330   A1=Cos(I6-1)
1340   A2=-Cos(N3-I6-1)
1350   FOR I7=0 TO N-I3 STEP I3
1360   I8=I7+I5-1
1370   I9=I8+I2
1380   T1=X(I8)
1390   T2=X(I9)
```

```
1400    T3=Y(I8)
1410    T4=Y(I9)
1420    A3=T1-T2
1430    A4=T3-T4
1440    X(I8)=T1+T2
1450    Y(I8)=T3+T4
1460    X(I9)=A1*A3-A2*A4
1470    Y(I9)=A1*A4+A2*A3
1480    NEXT I7
1490    NEXT I5
1500    NEXT I1
1510    I1=Log2n+1
1520    FOR I2=1 TO 14
1530    L(I2-1)=1
1540    IF I2>Log2n THEN 1560
1550    L(I2-1)=2^(I1-I2)
1560    NEXT I2
1570    K=0
1580    FOR I1=1 TO L(13)
1590    FOR I2=I1 TO L(12) STEP L(13)
1600    FOR I3=I2 TO L(11) STEP L(12)
1610    FOR I4=I3 TO L(10) STEP L(11)
1620    FOR I5=I4 TO L(9) STEP L(10)
1630    FOR I6=I5 TO L(8) STEP L(9)
1640    FOR I7=I6 TO L(7) STEP L(8)
1650    FOR I8=I7 TO L(6) STEP L(7)
1660    FOR I9=I8 TO L(5) STEP L(6)
1670    FOR I10=I9 TO L(4) STEP L(5)
1680    FOR I11=I10 TO L(3) STEP L(4)
1690    FOR I12=I11 TO L(2) STEP L(3)
1700    FOR I13=I12 TO L(1) STEP L(2)
1710    FOR I14=I13 TO L(0) STEP L(1)
1720    J=I14-1
1730    IF K>J THEN 1800
1740    A=X(K)
1750    X(K)=X(J)
1760    X(J)=A
1770    A=Y(K)
1780    Y(K)=Y(J)
1790    Y(J)=A
1800    K=K+1
1810    NEXT I14
1820    NEXT I13
1830    NEXT I12
1840    NEXT I11
1850    NEXT I10
1860    NEXT I9
1870    NEXT I8
1880    NEXT I7
1890    NEXT I6
1900    NEXT I5
1910    NEXT I4
1920    NEXT I3
1930    NEXT I2
1940    NEXT I1
1950    SUBEND
```

APPENDIX A.4 — EVALUATION OF WAVENUMBER INTENSITY SPECTRUM FOR ZERO DELAY ($\hat{t},\hat{t}' = 0$)

A program for the numerical evaluation of the zeroth-order Bessel transform of covariance $\hat{M}_y(\hat{\Delta R},0) - \hat{M}_y(\infty)$ is contained in this appendix. Inputs required of the user are listed in lines 10 - 40 and are coupled to appendix A.2, where the data input, AOR0 or AOR1 or AOR2, was generated. The numerical Bessel transform is accomplished by means of Simpson's rule with end correction [11; pages 414 - 418], and is exceedingly accurate for the small increment, .005, in $\hat{\Delta R}$ employed in line 30.

```
10    Dkc=.4                      !  INCREMENT IN k^
20    Nkc=200                     !  NUMBER OF k^ VALUES
30    Drc=.005                    !  INCREMENT IN DelR^
40    Nrc=900                     !  NUMBER OF DelR^ VALUES
50    DOUBLE Nrc,Nkc,I,Ns         !  INTEGERS
60    REDIM C(0:Nrc)
70    REDIM Wi(0:Nkc),W0(0:Nkc),W1(0:Nkc),W2(0:Nkc)
80    DIM C(900),Wi(200),W0(200),W1(200),W2(200)
90    ASSIGN #1 TO "AIR1"
100   READ #1;C(*)
110   FOR I=0 TO Nkc
120   Kc=I*Dkc                    !  k^
130   T=Kc*Drc
140   Se=So=0.
150   FOR Ns=1 TO Nrc-1 STEP 2
160   So=So+Ns*FNJo(T*Ns)*C(Ns)
170   NEXT Ns
180   FOR Ns=2 TO Nrc-2 STEP 2
190   Se=Se+Ns*FNJo(T*Ns)*C(Ns)
200   NEXT Ns
210   Wi(I)=C(0)+16.*So+14.*Se
220   NEXT I
230   MAT Wi=Wi*(Drc*Drc*2.*PI/15.)
240   ASSIGN #1 TO "AOR0"
250   READ #1;C(*)
260   FOR I=0 TO Nkc
270   Kc=I*Dkc
280   T=Kc*Drc
290   Se=So=0.
300   FOR Ns=1 TO Nrc-1 STEP 2
310   So=So+Ns*FNJo(T*Ns)*C(Ns)
320   NEXT Ns
330   FOR Ns=2 TO Nrc-2 STEP 2
340   Se=Se+Ns*FNJo(T*Ns)*C(Ns)
350   NEXT Ns
360   W0(I)=C(0)+16.*So+14.*Se
370   NEXT I
380   MAT W0=W0*(Drc*Drc*2.*PI/15.)
```

```
390   ASSIGN #1 TO "AOR1"
400   READ #1;C(*)
410   FOR I=0 TO Nkc
420   Kc=I*Dkc
430   T=Kc*Drc
440   Se=So=0.
450   FOR Ns=1 TO Nrc-1 STEP 2
460   So=So+Ns*FNJo(T*Ns)*C(Ns)
470   NEXT Ns
480   FOR Ns=2 TO Nrc-2 STEP 2
490   Se=Se+Ns*FNJo(T*Ns)*C(Ns)
500   NEXT Ns
510   W1(I)=C(0)+16.*So+14.*Se
520   NEXT I
530   MAT W1=W1*(Drc*Drc*2.*PI/15.)
540   ASSIGN #1 TO "AOR2"
550   READ #1;C(*)
560   ASSIGN #1 TO *
570   FOR I=0 TO Nkc
580   Kc=I*Dkc
590   T=Kc*Drc
600   Se=So=0.
610   FOR Ns=1 TO Nrc-1 STEP 2
620   So=So+Ns*FNJo(T*Ns)*C(Ns)
630   NEXT Ns
640   FOR Ns=2 TO Nrc-2 STEP 2
650   Se=Se+Ns*FNJo(T*Ns)*C(Ns)
660   NEXT Ns
670   W2(I)=C(0)+16.*So+14.*Se
680   NEXT I
690   MAT W2=W2*(Drc*Drc*2.*PI/15.)
700   GINIT 200/260
710   PLOTTER IS 505,"HPGL"
720   PRINTER IS 505
730   LIMIT PLOTTER 505,0,200,0,260
740   VIEWPORT 22,85,19,122
750   WINDOW 0,Nkc,-9,1
760      PRINT "VS5"
770   GRID 25,1
780      PRINT "VS36"
790   FOR I=0 TO Nkc
800   W=Wi(I)
810   IF W>0. THEN 840
820   PENUP
830   GOTO 850
840   PLOT I,LGT(W)
850   NEXT I
860   PENUP
```

```
870     FOR I=0 TO Nkc
880     W=W0(I)
890     IF W>0. THEN 920
900     PENUP
910     GOTO 930
920     PLOT I,LGT(W)
930     NEXT I
940     PENUP
950     FOR I=0 TO Nkc
960     W=W1(I)
970     IF W>0. THEN 1000
980     PENUP
990     GOTO 1010
1000    PLOT I,LGT(W)
1010    NEXT I
1020    PENUP
1030    FOR I=0 TO Nkc
1040    W=W2(I)
1050    IF W>0. THEN 1080
1060    PENUP
1070    GOTO 1090
1080    PLOT I,LGT(W)
1090    NEXT I
1100    PENUP
1110    PAUSE
1120    PRINTER IS CRT
1130    PLOTTER 505 IS TERMINATED
1140    END
1150    !
1160    DEF FNJo(X)          !  Jo(X) FOR ALL X
1170    Y=ABS(X)
1180    IF Y>8. THEN 1280
1190    T=Y*Y                !  HART, #5845
1200    P=2271490439.5536033-T*(5513584.5647707522-T*5292.6171303845574)
1210    P=2334489171877869.7-T*(47765559442673.588-T*(462172225031.71803-T*P))
1220    P=185962317621897804.-T*(44145829391815982.-|*P)
1230    Q=204251483.52134357+T*(494030.79491813972+T*(884.72036756175504+T))
1240    Q=2344750013658996.8+T*(15015462449769.752+T*(64398674535.133256+T*Q))
1250    Q=185962317621897733.+T*Q
1260    Jo=P/Q
1270    RETURN Jo
1280    Z=8./Y               !  HART, #6546 & 6946
1290    T=Z*Z
1300    Pn=2204.5010439651804+T*(128.67758574871419+T*.90047934748028803)
1310    Pn=8554.8225415066617+T*(8894.4375329606194+T*Pn)
1320    Pd=2214.0488519147104+T*(130.88490049992388+T)
1330    Pd=8554.8225415066628+T*(8903.8361417095954+T*Pd)
1340    Qn=13.990976865960680+T*(1.0497327982345548+T*.0093525953294031 9)
1350    Qn=-37.510534954957112-T*(46.093826814625175+T*Qn)
1360    Qd=921.56697552653090+T*(74.428389741411179+T)
1370    Qd=2400.6742371172675+T*(2971.9837452084920+T*Qd)
1380    T=Y-.78539816339744828
1390    Jo=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd-SIN(T)*Z*Qn/Qd)
1400    RETURN Jo
1410    FNEND
```

APPENDIX A.5 – EVALUATION OF PHASE MODULATION INTENSITY SPECTRUM

The normalized covariance function for phase modulation is given by (2.50) in the main text, namely

$$k_o(\zeta) = \exp\Big[-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)] - A[2 - \rho(\zeta)] + \qquad (A.5\text{-}1)$$
$$+ 2A [1 - \rho(\zeta)] \exp\left(-\mu_P^2/A\right) + A \rho(\zeta) \exp\left(-\frac{\mu_P^2}{A} [1 - \exp(-b\zeta)]\right)\Big]$$

for $\zeta \geq 0$, where ζ is the time delay and $\rho(\zeta)$ is the temporal normalized covariance of the field process. Also $\mu_P^2 = \mu_{PG}^2$. Since (A.5-1) involves an exponential of an exponential of an exponential, and because a wide range of parameter values are of interest, care must be taken in numerical evaluation of this covariance and its transform.

Observe first that

$$k_o(0) = 1 \quad \text{since } \rho(0) = 1 . \qquad (A.5\text{-}2)$$

Also, as delay $\zeta \to +\infty$, then $\rho \to 0$, giving

$$k_o(\infty) = \exp\left[-\Gamma'_A \mu_P^2 - 2A + 2A \exp\left(-\mu_P^2/A\right)\right] \neq 0 . \qquad (A.5\text{-}3)$$

The spectrum of interest is given by

$$W_o(\omega) = 4 \int_0^\infty d\zeta \cos(\omega\zeta)\, k_o(\zeta) \quad \text{for } \omega \geq 0 ; \quad \omega = 2\pi f . \qquad (A.5\text{-}4)$$

The nonzero value of (A.5-3) at $\zeta = \infty$ leads to an impulse in spectrum $W_o(\omega)$ at $\omega = 0$. This limiting value, $k_o(\infty)$, must be subtracted from covariance (A.5-1) prior to the numerical Fourier transform indicated by (A.5-4).

For $\Gamma_A' \mu_P^2 << 1$, the term

$$\exp\left(- \Gamma_A' \mu_P^2 [1 - \exp(-\zeta)]\right) \tag{A.5-5}$$

approaches its limiting value at $\zeta = +\infty$ as follows:

$$\begin{aligned} &\exp\left(- \Gamma_A' \mu_P^2 [1 - \exp(-\zeta)]\right) - \exp\left(- \Gamma_A' \mu_P^2\right) = \\ &= \exp\left(- \Gamma_A' \mu_P^2\right) \left[\exp\left(\Gamma_A' \mu_P^2 \exp(-\zeta)\right) - 1\right] = \\ &\sim \exp\left(- \Gamma_A' \mu_P^2\right) \Gamma_A' \mu_P^2 \exp(-\zeta) \ . \end{aligned} \tag{A.5-6}$$

This is a fairly rapid decay with ζ and will not lead to numerical difficulty when $\Gamma_A' \mu_P^2 << 1$.

For large $b\mu_P^2/A$, the term

$$\exp\left(- \frac{\mu_P^2}{A} [1 - \exp(-b\zeta)]\right) \tag{A.5-7}$$

is very sharp near $\zeta = 0$; in fact, it is given approximately by

$$\exp\left(- \frac{\mu_P^2}{A} b\zeta\right) \quad \text{for } \zeta \text{ near } 0 \ . \tag{A.5-8}$$

Therefore, we define the sharp component of covariance $k_o(\zeta)$ as

$$k_s(\zeta) = \exp\left[- A + A \exp\left(- \frac{\mu_P^2}{A} b\zeta\right)\right] - \exp(-A) \quad \text{for all } \zeta \ . \tag{A.5-9}$$

Then

$$k_s(0) = 1 - \exp(-A) \ , \qquad k_s(\infty) = 0 \ . \tag{A.5-10}$$

Now we let

$$k_o(\zeta) = [k_o(\zeta) - k_s(\zeta)] + k_s(\zeta) =$$

$$\equiv k_f(\zeta) + k_s(\zeta) \; , \qquad \text{(A.5-11)}$$

where $k_f(\zeta)$ is a flat function near $\zeta = 0$. Then we can express the desired difference as

$$k_o(\zeta) - k_o(\infty) = [k_f(\zeta) - k_o(\infty)] + k_s(\zeta) =$$

$$\equiv k_1(\zeta) + k_s(\zeta) \; , \qquad \text{(A.5-12)}$$

where functions $k_1(\zeta)$ and $k_s(\zeta)$ both decay to 0 at $\zeta = \infty$. We now employ two separate FFTs on each of the functions in (A.5-12). The sharp component, $k_s(\zeta)$, must be sampled with a very small increment, $\Delta\zeta$, when $b\mu_p^2/A$ is large. On the other hand, the flat component

$$k_1(\zeta) = k_f(\zeta) - k_o(\infty) \qquad \text{(A.5-13)}$$

can be sampled in a coarser fashion. Finally, if $b\mu_p^2/A$ is moderate, we work directly with $k_o(\zeta) - k_o(\infty)$ without breaking it into any components.

Two programs are furnished in this appendix, one for moderate $b\mu_p^2/A$, and the other for the flat component (A.5-13) when $b\mu_p^2/A$ is large. For sake of brevity, the Fourier transform of the sharp component (A.5-9) is straightforward and is not presented. The particular covariance $\rho(\zeta)$ adopted is triangular,

$$\rho(\zeta) = 1 - \frac{|\zeta|}{\zeta_c} \quad \text{for } |\zeta| < \zeta_c \; , \quad 0 \text{ otherwise} \; , \qquad \text{(A.5-14)}$$

but can easily be replaced. The parameter ζ_c is the cutoff value of covariance $\rho(\zeta)$.

The number of samples, N, taken of the covariance, in order to perform the FFT of (A.5-4), is rather large, so as to guarantee a very small value of truncation error at the upper end of the integral, despite the small increment $\Delta\zeta$. In order to keep the FFT size, Mf, at reasonable values, the data sequence is collapsed without any loss of accuracy according to the method given in [8; pages 7 - 8] and [9; pages 13 - 16]. The trapezoidal rule is used to approximate the integral in (A.5-4), for reasons given in [8; appendix A].

```
10  ! SPECTRUM FOR PHASE MODULATION - MODERATE
20    Mup=1.                    !  MUsubP
30    Gp=.001                   !  Gamma'
40    Bs=1.                     !  b
50    A=.2                      !  A
60    Zc=2.*PI                  !  Rho(Z) = 0 for |Z|>Zc; Z=zeta
70    Delz=.005                 !  Zeta increment
80    N=60000                   !  Maximum number of samples of ko(zeta)
90    Mf=16384                  !  Size of FFT
100   DOUBLE N,Mf,Ms,Ns         !  INTEGERS
110   DIM X(16384),Y(16384),Cos(4096)
120   REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130   MAT X=(0.)
140   MAT Y=(0.)
150   T=2.*PI/Mf
160   FOR Ms=0 TO Mf/4
170   Cos(Ms)=COS(T*Ms)         !  QUARTER-COSINE TABLE
180   NEXT Ms
190   Ta=Gp*Mup*Mup
200   IF A=0. THEN 220
210   Tb=Mup*Mup/A
220   Tc=2.*A*FNExp(Tb)
230   Kinf=FNExp(Ta+2.*A-Tc)  !  CORRELATION AT INFINITY
240   COM A,Bs,Zc,Ta,Tb,Tc,Kinf
250   T=1.-Kinf
260     PRINT 0,T
270   X(0)=T*.5                 !  TRAPEZOIDAL RULE
280   FOR Ns=1 TO N
290   Corr=FNKo(Ns*Delz)        !  CORRELATION ko(zeta)
300     IF Ns<6 THEN PRINT Ns,Corr
310   IF ABS(Corr)<1.E-30 THEN 350
320   Ms=Ns MODULO Mf           !  COLLAPSING
330   X(Ms)=X(Ms)+Corr
340   NEXT Ns
350   PRINT "Final value of Corr =";Corr;"  Ns =";Ns
360   MAT X=X*(Delz*4.)
370   CALL Fft14(Mf,Cos(*),X(*),Y(*))
```

```
380    GINIT
390    PLOTTER IS "GRAPHICS"
400    GRAPHICS ON
410    WINDOW -2,2,-60,0
420    LINE TYPE 3
430    GRID 1,10
440    LINE TYPE 1
450    Delf=1./(Mf*Delz)
460    FOR Ms=1 TO Mf/2
470    F=Ms*Delf                !  FREQUENCY
480    PLOT LGT(F),10.*LGT(X(Ms))
490    NEXT Ms
500    PENUP
510    PAUSE
520    END
530    !
540    DEF FNExp(Xminus)        !  EXP(-X) WITHOUT UNDERFLOW
550    IF Xminus>708.3 THEN RETURN 0.
560    RETURN EXP(-Xminus)
570    FNEND
580    !
590    DEF FNKo(Zeta)           !  CORRELATION ko(zeta)
600    COM A,Bs,Zc,Ta,Tb,Tc,Kinf
610    Rho=MAX(0.,1.-Zeta/Zc)   !  TRIANGULAR RHO
620    E1=Ta*(1.-FNExp(Zeta))
630    E2=Tb*(1.-FNExp(Bs*Zeta))
640    E3=A*Rho*FNExp(E2)
650    RETURN FNExp(E1+A*(2.-Rho)-Tc*(1.-Rho)-E3)-Kinf
660    FNEND
670    !
680    SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
690    ! SEE APPENDIX A.3
```

```
 10  ! SPECTRUM FOR PHASE MODULATION - FLAT COMPONENT
 20    Mup=1.                   !  MUsubP
 30    Gp=.001                  !  Gamma'
 40    Bs=1.                    !  b
 50    A=0.                     !  A
 60    Zc=2.*PI                 !  Rho(Z) = 0 for |Z|>Zc; Z=zeta
 70    Delz=.005                !  Zeta increment
 80    N=60000                  !  Maximum number of samples of k1(zeta)
 90    Mf=16384                 !  Size of FFT
100    DOUBLE N,Mf,Ms,Ns        !  INTEGERS
110    DIM X(16384),Y(16384),Cos(4096)
120    REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130    MAT X=(0.)
140    MAT Y=(0.)
150    T=2.*PI/Mf
160    FOR Ms=0 TO Mf/4
170    Cos(Ms)=COS(T*Ms)        !  QUARTER-COSINE TABLE
180    NEXT Ms
190    Ta=Gp*Mup*Mup
200    IF A=0. THEN 220
```

```
210     Tb=Mup*Mup/A
220     Tc=2.*A*FNExp(Tb)
230       Tb=5.E55
240     Kinf=FNExp(Ta+2.*A-Tc)  !  CORRELATION AT INFINITY
250     Ea=FNExp(A)
260     Tbb=Tb*Bs
270     COM A,Bs,Zc,Ta,Tb,Tc,Kinf,Ea,Tbb
280     T=1.-Kinf-(1.-Ea)       !  SUBTRACT SHARP COMPONENT
290       PRINT 0,T
300     X(0)=T*.5               !  TRAPEZOIDAL RULE
310     FOR Ns=1 TO N
320     Corr=FNK1(Ns*Delz)      !  CORRELATION k1(zeta)
330       IF Ns<6 THEN PRINT Ns,Corr
340     IF ABS(Corr)<1.E-30 THEN 380
350     Ms=Ns MODULO Mf         !  COLLAPSING
360     X(Ms)=X(Ms)+Corr
370     NEXT Ns
380     PRINT "Final value of Corr =";Corr;"  Ns =";Ns
390     MAT X=X*(Delz*4.)
400     CALL Fft14(Mf,Cos(*),X(*),Y(*))
410     GINIT
420     PLOTTER IS "GRAPHICS"
430     GRAPHICS ON
440     WINDOW -2,2,-60,0
450     LINE TYPE 3
460     GRID 1,10
470     LINE TYPE 1
480     Delf=1./(Mf*Delz)
490     FOR Ms=1 TO Mf/2
500     F=Ms*Delf               !  FREQUENCY
510     T=X(Ms)
520     IF T>0. THEN 550
530     PENUP
540     GOTO 560
550     PLOT LGT(F),10.*LGT(T)
560     NEXT Ms
570     PENUP
580     PAUSE
590     END
600     !
610     DEF FNExp(Xminus)       !  EXP(-X) WITHOUT UNDERFLOW
620     IF Xminus>708.3 THEN RETURN 0.
630     RETURN EXP(-Xminus)
640     FNEND
650     !
660     DEF FNK1(Zeta)          !  CORRELATION k'(zeta)
670     COM A,Bs,Zc,Ta,Tb,Tc,Kinf,Ea,Tbb
680     Rho=MAX(0.,1.-Zeta/Zc)  !  TRIANGULAR RHO
690     E1=Ta*(1.-FNExp(Zeta))
700     E2=Tb*(1.-FNExp(Bs*Zeta))
710     E3=A*Rho*FNExp(E2)
720     E4=FNExp(Tbb*Zeta)
730     Sharp=FNExp(A*(1.-E4))-Ea     !  ks(zeta)
740     RETURN FNExp(E1+A*(2.-Rho)-Tc*(1.-Rho)-E3)-Kinf-Sharp
750     FNEND
760     !
770     SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
780     ! SEE APPENDIX A.3
```

APPENDIX A.6 – EVALUATION OF FREQUENCY MODULATION INTENSITY SPECTRUM

The normalized covariance function for frequency modulation is given by (2.48) in the main text, namely

$$k_o(\zeta) = \exp\Bigg[- \Gamma_A' \, \mu_F^2 \, [\exp(-\zeta) + \zeta - 1] - A[2 - \rho(\zeta)] + \\ + A \, \rho(\zeta) \, \exp\bigg(- \frac{\mu_F^2}{Ab^2} \, [\exp(-b\zeta) + b\zeta - 1]\bigg)\Bigg] \quad \text{for } \zeta \geq 0 \; , \qquad \text{(A.6-1)}$$

where ζ is the time delay and $\rho(\zeta)$ is the temporal normalized covariance of the field process. Also, $\mu_F^2 = \mu_{FG}^2$ and $b = \Delta\omega_A/\Delta\omega_N$. Since (A.6-1) involves an exponential of an exponential of an exponential, and because a wide range of parameter values are of interest, care must be taken in numerical evaluation of this covariance and its transform.

Observe that

$$k_o(0) = 1 \; , \quad \text{because } \rho(0) = 1 \; . \qquad \text{(A.6-2)}$$

Also, as delay $\zeta \to +\infty$, then $\rho \to 0$, giving

$$k_o(\zeta) \sim \exp\left(- \Gamma_A' \, \mu_F^2 \, (\zeta - 1) - 2A\right) \equiv k_1(\zeta) \text{ for } \zeta > 0 \; . \qquad \text{(A.6-3)}$$

This term, $k_1(\zeta)$, decays slowly with ζ if $\Gamma_A' \, \mu_F^2 << 1$.

The spectrum of interest is given by

$$W_o(\omega) = 4 \int_0^\infty d\zeta \, \cos(\omega\zeta) \, k_o(\zeta) \text{ for } \omega \geq 0 \; ; \quad \omega = 2\pi f \; . \qquad \text{(A.6-4)}$$

The spectrum corresponding to the limiting component, $k_1(\zeta)$ in (A.6-3), is directly available in closed form as

$$W_1(\omega) = 4 \int_0^\infty d\zeta \cos(\omega\zeta)\, k_1(\zeta) =$$

$$= \exp\left(\Gamma_A' \mu_F^2 - 2A\right) \frac{4\, \Gamma_A' \mu_F^2}{\left(\Gamma_A' \mu_F^2\right)^2 + \omega^2} \; . \tag{A.6-5}$$

If $\Gamma_A' \mu_F^2 << 1$, this latter quantity is large and very sharply peaked at $\omega = 0$; hence, this term has been subtracted out and handled separately when $\Gamma_A' \mu_F^2 << 1$. The residual covariance, $k_o(\zeta) - k_1(\zeta)$, then decays very rapidly with ζ and is easily handled directly by means of an FFT. This breakdown is not necessary when $\Gamma_A' \mu_F^2 \sim 1$ and is avoided, then, by handling $k_o(\zeta)$ directly in one FFT.

For $\mu_F^2/A >> 1$, the term

$$\exp\left(- \frac{\mu_F^2}{Ab^2} [\exp(-b\zeta) + b\zeta - 1]\right) \tag{A.6-6}$$

inside the exponential in (A.6-1) behaves like

$$\exp\left(- \frac{\mu_F^2}{A} \frac{1}{2} \zeta^2\right) \quad \text{near } \zeta = 0 \; , \tag{A.6-7}$$

where its major sharp contribution arises. For example, if $\mu_F = 50$, $A = 1$, then increment $\Delta\zeta = .005$ leads to values for (A.6-7) of

$$\exp(-0.156\, n^2) \text{ at } \zeta = n\, \Delta\zeta \; , \tag{A.6-8}$$

which is adequately sampled in order to track its dominant contribution; the actual sequence of values is 1, .856, .536, .247, .083. For smaller μ_F^2/A, this sampling interval is more than adequate.

Two programs are furnished in this appendix, one each for the cases of large and small $\Gamma_A' \mu_F^2$. The particular covariance $\rho(\zeta)$ adopted is triangular,

$$\rho(\zeta) = 1 - \frac{|\zeta|}{\zeta_c} \quad \text{for } |\zeta| < \zeta_c \text{ , } \quad 0 \text{ otherwise ,} \qquad \text{(A.6-9)}$$

but can easily be replaced. The parameter ζ_c is the cutoff covariance value and is specified numerically in the examples in figures 3.9a through 3.10b.

```
10  ! SPECTRUM FOR FREQUENCY MODULATION - LARGE Gp*Muf^2
20    Muf=50.                      !  MUsubF
30    Gp=.001                      !  Gamma'
40    Bs=1.                        !  b
50    A=.2                         !  A
60    Zc=2.*PI                     !  Rho(Z) = 0 for |Z|>Zc; Z=zeta
70    Delz=.005                    !  Zeta increment
80    N=60000                      !  Maximum number of samples of ko(zeta)
90    Mf=16384                     !  Size of FFT
100   DOUBLE N,Mf,Ms,Ns            !  INTEGERS
110   DIM X(16384),Y(16384),Cos(4096)
120   REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130   MAT X=(0.)
140   MAT Y=(0.)
150   T=2.*PI/Mf
160   FOR Ms=0 TO Mf/4
170   Cos(Ms)=COS(T*Ms)            !  QUARTER-COSINE TABLE
180   NEXT Ms
190   Ta=Gp*Muf*Muf
200   IF A=0. THEN 220
210   Tb=Muf*Muf/(A*Bs*Bs)
220   Tc=FNExp(2.*A-Ta)*Ta
230   Td=Ta*Ta
240   COM A,Bs,Zc,Ta,Tb
250   X(0)=.5                      !  TRAPEZOIDAL RULE
260   FOR Ns=1 TO N
270   Corr=FNKo(Ns*Delz)           !  CORRELATION ko(zeta)
280   IF Corr<1.E-20 THEN 320
290   Ms=Ns MODULO Mf              !  COLLAPSING
300   X(Ms)=X(Ms)+Corr
310   NEXT Ns
320   PRINT "Final value of Corr =";Corr;"  Ns =";Ns
330   MAT X=X*(Delz)
340   CALL Fft14(Mf,Cos(*),X(*),Y(*))
```

```
350     GINIT
360     PLOTTER IS "GRAPHICS"
370     GRAPHICS ON
380     WINDOW -4,2,-70,30
390     LINE TYPE 3
400     GRID 1,10
410     LINE TYPE 1
420     Delf=1./(Mf*Delz)
430     FOR Ms=1 TO Mf/2
440     F=Ms*Delf                   !  FREQUENCY
450     T=X(Ms)
460     IF T>0. THEN 490
470     PENUP
480     GOTO 500
490     PLOT LGT(F),10.*LGT(T)
500     NEXT Ms
510     PENUP
520     Add=X(0)-Tc/Td              !  ORIGIN CORRECTION
530     F=1.E-4
540     FOR Ns=1 TO 100
550     W=2.*PI*F
560     W1=Tc/(Td+W*W)
570     T=W1+Add
580     IF T>0. THEN 610
590     PENUP
600     GOTO 620
610     PLOT LGT(F),10.*LGT(W1+Add)
620     F=F*1.1                     !  FREQUENCY
630     IF F>Delf THEN 650
640     NEXT Ns
650     PENUP
660     PAUSE
670     END
680     !
690     DEF FNExp(Xminus)           !  EXP(-X) WITHOUT UNDERFLOW
700     IF Xminus>708.3 THEN RETURN 0.
710     RETURN EXP(-Xminus)
720     FNEND
730     !
740     DEF FNKo(Zeta)              !  CORRELATION ko(zeta)
750     COM A,Bs,Zc,Ta,Tb
760     E1=FNExp(Zeta)+Zeta-1.
770     T=Bs*Zeta
780     E2=FNExp(T)+T-1.
790     Rho=MAX(0.,1-Zeta/Zc)    !  TRIANGULAR RHO
800     T=Ta*E1+A*(2.-Rho)-A*Rho*FNExp(Tb*E2)
810     RETURN FNExp(T)
820     FNEND
830     !
840     SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
850     ! SEE APPENDIX A.3
```

```
10   ! SPECTRUM FOR FREQUENCY MODULATION - SMALL Gp*Muf^2
20     Muf=1.                      !  MUsubF
30     Gp=.001                     !  Gamma'
40     Bs=1.                       !  b
50     A=.2                        !  A
60     Zc=2.*PI                    !  Rho(Z) = 0 for |Z|>Zc; Z=zeta
70     Delz=.005                   !  Zeta increment
80     N=10000                     !  Maximum number of samples of ko(zeta)
90     Mf=8192                     !  Size of FFT
100    DOUBLE N,Mf,Ms,Ns           !  INTEGERS
110    DIM X(8192),Y(8192),Cos(2048)
120    REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130    MAT X=(0.)
140    MAT Y=(0.)
150    T=2.*PI/Mf
160    FOR Ms=0 TO Mf/4
170    Cos(Ms)=COS(T*Ms)           !  QUARTER-COSINE TABLE
180    NEXT Ms
190    Ta=Gp*Muf*Muf
200    IF A=0. THEN 220
210    Tb=Muf*Muf/(A*Bs*Bs)
220    T=FNExp(2.*A-Ta)
230    Tc=T*Ta
240    Td=Ta*Ta
250    Delf=.1*Ta/(2.*PI)          !  INCREMENT IN FREQUENCY
260    COM A,Bs,Zc,Ta,Tb
270    X(0)=.5*(1.-T)              !  TRAPEZOIDAL RULE
280    FOR Ns=1 TO N
290    Corr=FNKo1(Ns*Delz)         !  CORRELATION ko(zeta)-k1(zeta)
300    IF ABS(Corr)<1.E-30 THEN 340
310    Ms=Ns MODULO Mf             !  COLLAPSING
320    X(Ms)=X(Ms)+Corr
330    NEXT Ns
340    PRINT "Final value of Corr =";Corr;"   Ns =";Ns
350    MAT X=X*(Delz)
360    CALL Fft14(Mf,Cos(*),X(*),Y(*))
```

```
370     GINIT
380     PLOTTER IS "GRAPHICS"
390     GRAPHICS ON
400     WINDOW -4,2,-70,30
410     LINE TYPE 3
420     GRID 1,10
430     LINE TYPE 1
440     FOR Ms=1 TO 2000
450     F=Ms*Delf                 !  FREQUENCY
460     W=2.*PI*F
470     W1=Tc/(Td+W*W)            !  SHARP SPECTRAL COMPONENT
480     T=Mf*Delz*F
490     Ns=INT(T)
500     Fr=T-Ns
510     W2=Fr*X(Ns+1)+(1.-Fr)*X(Ns)    !  BROAD SPECTRAL COMPONENT
520     PLOT LGT(F),10.*LGT(W1+W2)
530     NEXT Ms
540     Ns=MAX(Ns,1)
550     FOR Ms=Ns TO Mf/2
560     F=Ms/(Mf*Delz)            !  FREQUENCY
570     W=2.*PI*F
580     W1=Tc/(Td+W*W)
590     W2=X(Ms)
600     T=W1+W2
610     IF T>0. THEN 640
620     PENUP
630     GOTO 650
640     PLOT LGT(F),10.*LGT(T)
650     NEXT Ms
660     PENUP
670     PAUSE
680     END
690     !
700     DEF FNExp(Xminus)         !  EXP(-X) WITHOUT UNDERFLOW
710     IF Xminus>708.3 THEN RETURN 0.
720     RETURN EXP(-Xminus)
730     FNEND
740     !
750     DEF FNKo1(Zeta)           !  CORRELATION ko(zeta)-k1(zeta)
760     COM A,Bs,Zc,Ta,Tb
770     E1=FNExp(Zeta)+Zeta-1.
780     T=Bs*Zeta
790     E2=FNExp(T)+T-1.
800     Rho=MAX(0.,1-Zeta/Zc)     !  TRIANGULAR RHO
810     T=Ta*E1+A*(2.-Rho)-A*Rho*FNExp(Tb*E2)
820     RETURN FNExp(T)-FNExp(Ta*(Zeta-1.)+2.*A)
830     FNEND
840     !
850     SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
860     ! SEE APPENDIX A.3
```

REFERENCES

[1] D. Middleton, **An Introduction to Statistical Communication Theory**, McGraw-Hill Book Company, Inc., New York, NY, 1960. Also, Reprint Edition, Peninsula Publishing Co., P.O. Box 867, Los Altos, CA, 1987.

[2] D. Middleton, "Second-Order Non-Gaussian Probability Distributions and Their Applications to "Classical" Nonlinear Signal Processing Problems in Communication Theory," to be submitted to **IEEE Transactions on Information Theory**, 1991. (This is a considerably expanded version of [3] ff.)

[3] D. Middleton, "Second-Order Non-Gaussian Probability Distributions and Their Applications to Classical Nonlinear Processing Problems," **Proceedings of 20th Conference on Information Sciences and Systems**, pages 394 - 400, March 19 - 21, 1986, Princeton University, NJ.

[4] D. Middleton, **First- and Second-Order Gauss-Composite Processes, with Applications to Ambient and Scatter Noise**, in preparation, 1991.

[5] D. Middleton, "Canonical and Quasi-Canonical Probability Models of Class A Interference," **IEEE Transactions on Electromagnetic Compatibility**, volume EMC-25, number 2, May 1983.

[6] A. H. Nuttall, I. B. Cohen, and D. Middleton, **Performance Parameters for Quasi-Canonical Class A Non-Gaussian Noise; Source Distribution Law $\mu = 0$, Propagation Law $\gamma = 2$**, NUSC Technical Report 7715, Naval Underwater Systems Center, New London, CT, 15 June 1986.

[7] D. Middleton, "Channel Modeling and Threshold Signal Processing in Underwater Acoustics: An Analytical Overview," **IEEE Journal Oceanic Engineering**, volume OE-12, number 1, pages 4 - 28, January 1987.

[8] A. H. Nuttall, **Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions**, NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

[9] A. H. Nuttall, "Alternative Forms and Computational Considerations for Numerical Evaluation of Cumulative Probability Distributions Directly from Characteristic Functions," **Proceedings IEEE**, volume 58, number 11, pages 1872 - 1873, November 1970. Also in NUSC Report Number NL-3012, Naval Underwater Systems Center, New London, CT, 12 August 1970.

[10] **Handbook of Mathematical Functions**, U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series number 55, U. S. Government Printing Office, Washington, DC, June 1964.

[11] C. Lanczos, **Applied Analysis**, third printing, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[12] J. A. Mullen and D. Middleton, "The Rectification of Non-Gaussian Noise," **Quarterly Journal of Applied Mathematics**, volume XV, number 4, pages 395 - 419, January 1958.

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29 July 1991

Exact Performance of Filtered and Weighted Energy Detector with Mismatched Frequency and Time Locations and Characteristics

Albert H. Nuttall

ABSTRACT

The characteristic function of the output of a filtered and weighted energy detector, subjected to a nonstationary Gaussian input signal in the presence of colored Gaussian noise, is derived in closed form. This result allows and accounts for mismatch in the receiving filter frequency characteristics relative to the received signal and noise spectra. It also allows for uncertainty in the time location and/or duration of the received signal, by way of arbitrary time weighting of the squared filter output.

Programs for evaluation of the receiver operating characteristics, for white noise inputs as well as colored noise inputs, are furnished in BASIC and exercised for a number of examples. This approach allows for precise quantitative investigation of the degradation associated with filtered energy detection in the presence of uncertainties.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Δ	time sampling increment, figure 1
k,m	integers denoting discrete time, figure 1 and (1)
$x(k\Delta)$	digital input sequence, figure 1 and (1)
$h(m\Delta)$	digital filter, figure 1 and (5)
$y(k\Delta)$	filter output sequence, figure 1 and (5)
$w(k\Delta)$	summer weights, figure 1, (6) and (7)
z	system output, figure 1 and (6)
$n(k\Delta)$	input noise sequence, (1)
$s(k\Delta)$	input signal sequence, (1)
$g(k\Delta)$	signal gating sequence, (1) and (4)
overbar	ensemble average, (2)
$R_n(m\Delta)$	input noise covariance, (2)
$R_s(m\Delta)$	input signal covariance, (3)
k_a	input signal start time integer, (4)
k_b	input signal stop time integer, (4)
$y_s(k\Delta)$	filter output signal, (5) and (13)
$y_n(k\Delta)$	filter output noise, (5) and (8)
k_c	observation start time integer, (7)
k_d	observation stop time integer, (7)
$k_1\Delta, k_2\Delta$	general covariance times, (9)
C_n	filter output noise covariance, (9)
$\phi_h(j\Delta)$	filter correlation function, (10)
σ_n^2	input noise power, (11) and (45)
C_s	filter output signal covariance, (15)
K_1, K_2	auxiliary integers, (18)

K_o	auxiliary integer, (19)
C_y	covariance of total filter output, (23)
a_k	auxiliary random variables, (24)
C_a	covariance of $\{a_k\}$, (26)
C	matrix with elements (26)
Q	normalized modal matrix of eigenvectors of C, (27)
Λ	diagonal matrix of eigenvalues $\{\lambda_k\}$ of C, (27)
T	transpose of matrix, (27)
A	column vector of $\{a_k\}$, (28)
B	linear transformation of A, (29)
$\{b_k\}$	elements of B, (30)
$f_z(\xi)$	characteristic function of output z, (32)
τ	argument (delay) of correlation functions, (33)
J	number of input signal covariance components, (33)
α_j	input signal scaling, (33)
τ_{sj}	input signal time constant, (33)
σ_s^2	input signal power, (34)
f	frequency, (35)
$G_s(f)$	input signal spectrum, (35)
ω	$2\pi f$, (35)
a_j	input signal exponential parameter, (36)
M	number of input noise covariance components, (44)
β_m	input noise scaling, (44)
τ_{nm}	input noise time constant, (44)
b_m	input noise exponential parameter, (46)

N number of filter components, (47)

ψ_n filter scaling, (47)

τ_{hn} filter time constant, (47)

c_n filter exponential parameter, (49)

ν_n auxiliary filter constant, (51)

$\underline{H}(z)$ filter z transform, (52)

$\circledast$ convolution, (55) and (A-2)

$C(k\Delta)$ auxiliary function, (56)

μ_m, γ_n auxiliary constants, (60)

S auxiliary function, (65) and (69)

g auxiliary function, (68)

$f_o(\xi)$ characteristic function for noise only, (73)

$p_o(u)$ probability density function for noise only, (73)

Δ_o sampling increment in ξ applied to $f_o(\xi)$, (73)

$\tilde{p}_o(u)$ approximation to $p_o(u)$, (73) and (74)

N_o size of FFT for $\tilde{p}_o(u)$, (74)

δ_o increment in u for $\tilde{p}_o(u)$, (74) and (75)

u_o maximum useful u for $\tilde{p}_o(u)$, (75)

$f_1(\xi)$ characteristic function for signal present, (76)

$p_1(u)$ probability density function for signal present, (76)

Δ_1 sampling increment in ξ applied to $f_1(\xi)$, (76)

$\tilde{p}_1(u)$ approximation to $p_1(u)$, (76) and (77)

N_1 size of FFT for $\tilde{p}_1(u)$, (77)

δ_1 increment in u for $\tilde{p}_1(u)$, (77) and (78)

u_1 maximum useful u for $\tilde{p}_1(u)$, (78)

$E_k(u)$ exceedance distribution corresponding to $f_k(\xi)$, (80)

Im imaginary part, (80)

P_D	detection probability
P_F	false alarm probability
CDF	cumulative distribution function
EDF	exceedance distribution function
R	input signal to noise ratio σ_s^2/σ_n^2, (82)
$\delta_b(\tau)$	unit impulse train with spacing b, (A-2)

EXACT PERFORMANCE OF FILTERED AND WEIGHTED ENERGY DETECTOR WITH MISMATCHED FREQUENCY AND TIME LOCATIONS AND CHARACTERISTICS

INTRODUCTION

The detection of weak random signals of unknown duration, bandwidth, time location, and frequency shift in the presence of colored noise is often accomplished in practice by filtering the received waveform to the frequency band of interest, squaring the filter output, and time-weighting this quantity prior to accumulation and threshold comparison. The performance of this processor obviously depends on the spectra of the received signal and noise processes as well as on the transfer characteristics of the filter employed. Lack of detailed knowledge of the signal spectral characteristics or frequency shift will often dictate that a fairly broad (mismatched) filter passband be utilized in order that the signal be passed when present. Similarly, uncertainty about the signal location or duration will mandate that the receiver observation time be lengthened in order to ensure capture of any impingent signal energy.

Additionally, the received signal process is often characterized as a finite-duration burst of a stationary process. Since the signal time location and duration are often unknown, the receiving filter must be kept open at all times, thereby allowing the filter output noise to reach its full-strength steady state value. By contrast, a gated input signal leads to a filter output component which gradually builds up in time and

then decays slowly after the signal is turned off. These nonstationary effects, coupled with the particular time weighting employed during the observation interval, influence the performance of the detector in a complicated nonobvious fashion.

There is a need to be able to calculate exactly the performance of this type of signal processor in the presence of such deleterious factors, so that the degradations associated with lack of knowledge can be ascertained quantitatively. Furthermore, it will not suffice to resort to Gaussian approximations for the processor output, because the number of samples employed are not large enough to utilize the central limit theorem, especially on the tails of the distribution (small false alarm probabilities) where we are interested. This need is addressed in this report for the case of Gaussian input signals in Gaussian noise, with the result that programs are furnished for exactly assessing the performance of the mismatched energy detector for a very wide variety of characteristics and selections of parameters. In particular, no presumptions are made about the input signal time-bandwidth product or about the size of the product of observation time and filter bandwidth.

SYSTEM DESCRIPTION AND ASSUMPTIONS

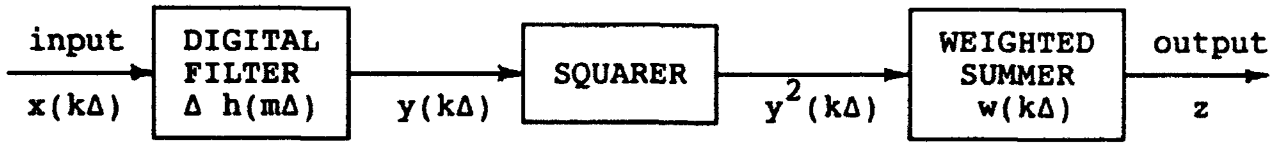

Figure 1. Block Diagram of Detector

The detector of interest is the digital processor depicted in figure 1; time sampling increment Δ is arbitrary but should be approximately matched to the coherence times of the input signal and filter. The digital input sequence $x(k\Delta)$, filter impulse response $\Delta\ h(m\Delta)$, and summer weights $w(k\Delta)$ are all real. Input $x(k\Delta)$ consists of either noise-alone or gated signal plus noise, according to model

$$x(k\Delta) = \left\{ \begin{array}{c} n(k\Delta) \\ \text{or} \\ s(k\Delta)\ g(k\Delta) + n(k\Delta) \end{array} \right\} \text{ for all } k . \tag{1}$$

Input noise $n(k\Delta)$ is present for all time and is a stationary zero-mean discrete Gaussian sequence with covariance

$$R_n(m\Delta) = \overline{n(k\Delta)\ n(k\Delta - m\Delta)} \quad \text{for all } m,k , \tag{2}$$

where an overbar denotes an ensemble average. The numerical evaluation of the values of the noise covariance, directly from a specified noise spectrum $G_n(f)$, is considered in appendix A.

Underlying input signal $s(k\Delta)$ in (1) (if present) is also a

stationary zero-mean discrete Gaussian sequence with covariance

$$R_s(m\Delta) = \overline{s(k\Delta)\ s(k\Delta - m\Delta)} \quad \text{for all } m,k \ . \tag{3}$$

However, input signal samples $s(k\Delta)$ are gated by function $g(k\Delta)$ which is nonzero only in a limited time region:

$$g(k\Delta) = 1 \quad \text{for } k_a \le k \le k_b \text{ , zero otherwise .} \tag{4}$$

This results in a gated burst of stationary signal sequence $s(k\Delta)$ being inputted to digital filter $h(m\Delta)$; the input signal starting and ending times $k_a\Delta$ and $k_b\Delta$, respectively, are generally unknown, except in an approximate way. This generality allows for consideration of input signals of unknown arrival time and duration at the detector input.

The filter output $y(k\Delta)$ can be conveniently broken into signal and noise components, in accordance with (1), and is available by means of discrete convolution

$$y(k\Delta) = \sum_m \Delta\ h(m\Delta)\ x(k\Delta - m\Delta) = y_s(k\Delta) + y_n(k\Delta) \ , \tag{5}$$

when an input signal is present. (Summations without limits are over $(-\infty,+\infty)$.) However, due to the gating in (1), filter output signal $y_s(k\Delta)$ will have transient phases, including a buildup just after time $k_a\Delta$ and a decay just after time $k_b\Delta$. This nonstationary behavior is included and exactly accounted for in the following analysis.

The filter output (5) is then squared, scaled by weights $w(k\Delta)$, and summed to give system output

$$z = \sum_k w(k\Delta)\ y^2(k\Delta)\ , \tag{6}$$

where the weights are nonzero only in a limited observation time specified by

$$w(k\Delta) \neq 0 \quad \text{for } k_c \le k \le k_d\ . \tag{7}$$

These weights need not be uniform; for example, they could be exponential. Output z is compared with a threshold for a declaration of signal absent versus signal present.

Observation time limits $k_c\Delta$ and $k_d\Delta$ are under the control of the receiver processor, and would hopefully match fairly well the time interval over which the gated input signal is received; see (4). But, in any event, the degree of generality in (7) allows for investigation of observation times that might be too short, thereby losing some signal energy, or too long, thereby picking up additional unwanted noise components; both situations lead to deterioration of the detectability of weak signals and should be avoided if possible. An illustration of the parameters is given in figure 2, for signal-only present.

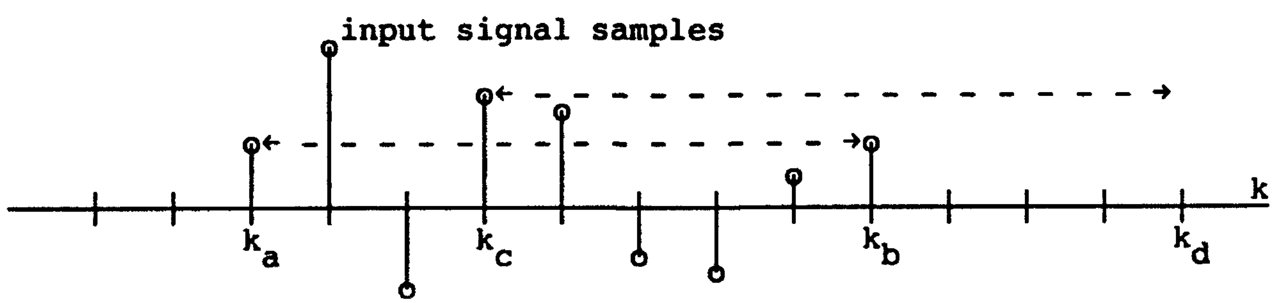

Figure 2. Gated and Observation Intervals; No Noise

FILTER OUTPUT NOISE CHARACTERIZATION

The filter output noise sequence is obtained from discrete convolution (5) and (1) as

$$y_n(k\Delta) = \Delta \sum_m h(m\Delta)\; n(k\Delta - m\Delta) \quad \text{for all } k \;, \tag{8}$$

where it is presumed that the filter $h(m\Delta)$ has been open to noise input $n(k\Delta)$ for all time; this results in noise output $y_n(k\Delta)$ being a zero-mean stationary Gaussian sequence. The filter output noise covariance at general times $k_1\Delta$ and $k_2\Delta$ is given by

$$C_n(k_1\Delta, k_2\Delta) = \overline{y_n(k_1\Delta)\; y_n(k_2\Delta)} =$$

$$= \Delta^2 \sum_{mp} h(m\Delta)\; h(p\Delta)\; \overline{n(k_1\Delta - m\Delta)\; n(k_2\Delta - p\Delta)} =$$

$$= \Delta^2 \sum_{mp} h(m\Delta)\; h(p\Delta)\; R_n(k_1\Delta - k_2\Delta + p\Delta - m\Delta) =$$

$$= \sum_j \phi_h(j\Delta)\; R_n(k_1\Delta - k_2\Delta - j\Delta) \;, \tag{9}$$

where the filter correlation function is defined as

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta)\; h(m\Delta - j\Delta) \quad \text{for all } j \;. \tag{10}$$

The right-hand side of (9) is a function only of the time difference $k_1\Delta - k_2\Delta$, in keeping with the stationarity of filter output noise $y_n(k\Delta)$.

If the input noise in figure 1 is white, its covariance in (2) becomes

$$R_n(m\Delta) = \sigma_n^2 \, \delta_{mo} \,, \tag{11}$$

where σ_n^2 is the input noise power. In this case, the filter output noise covariance in (9) simplifies to

$$C_n(k_1\Delta, k_2\Delta) = \sigma_n^2 \, \phi_h(k_1\Delta - k_2\Delta) \quad \text{for white input noise} \,. \tag{12}$$

The numerical evaluation of filter correlation function $\phi_h(j\Delta)$ in (10), directly from a specified transfer function $H(f)$, is considered in appendix B. A possible problem with discontinuous functions is treated in appendix C.

FILTER OUTPUT SIGNAL CHARACTERIZATION

The filter output signal sequence (when present) follows from (5) and (1) as

$$y_s(k\Delta) = \Delta \sum_m h(k\Delta - m\Delta)\ s(m\Delta)\ g(m\Delta) \quad \text{for all } k\ . \tag{13}$$

The finite duration of gating sequence $g(k\Delta)$ in (4), as well as the realizability of impulse response $\Delta\ h(m\Delta)$, will serve to terminate the summation in (13) at finite limits.

We presume that filter $h(m\Delta)$ in figure 1 is realizable; that is,

$$h(m\Delta) = 0 \quad \text{for } m < 0\ . \tag{14}$$

This makes filter output signal $y_s(k\Delta)$ in (13) equal to zero for $k < k_a$. On the other hand, if the filter has an infinite-duration impulse response $\Delta\ h(m\Delta)$, then $y_s(k\Delta)$ is nonzero for $k \geq k_a$; see figure 3. However, the filter output noise $y_n(k\Delta)$ will dominate the signal output for $k >> k_b$, meaning that performance of the detector will be poor in the case when $k_d >> k_b$. That is, too long an observation time in (7), relative to the signal duration, is detrimental to signal detectability.

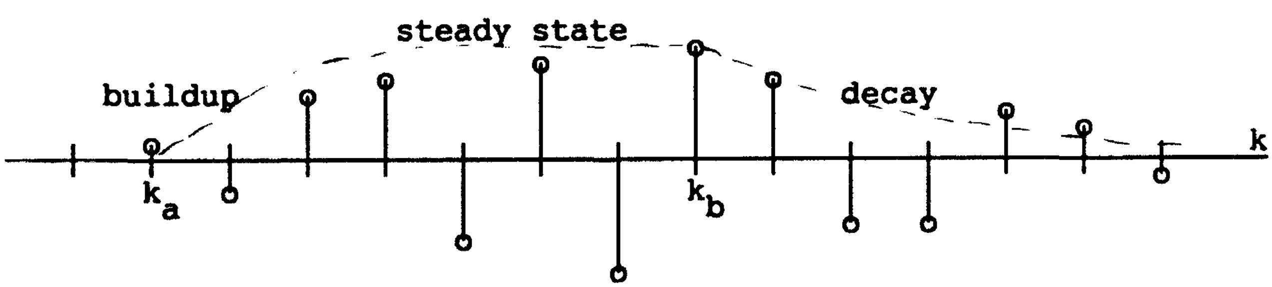

Figure 3. Filter Output Signal Buildup and Decay

The signal output, $y_s(k\Delta)$ in (13), from filter $h(m\Delta)$ is <u>not</u> stationary. The filter output signal covariance is defined as

$$C_s(k_1\Delta,k_2\Delta) = \overline{y_s(k_1\Delta)\ y_s(k_2\Delta)} = C_s(k_2\Delta,k_1\Delta) \quad \text{for all } k_1,k_2. \quad (15)$$

This function is zero for $k_1 < k_a$ or $k_2 < k_a$; therefore, in the following, we can confine calculation of (15) to $k_1 \geq k_a$ and $k_2 \geq k_a$. The particular filter output signal sample $y_s(k_a\Delta)$ is nonzero only if $h(0) \neq 0$.

Substitution of (13) in (15) and the use of (3) yields

$$C_s(k_1\Delta,k_2\Delta) = \Delta^2 \sum_{mp} h(k_1\Delta-m\Delta)\ h(k_2\Delta-p\Delta)\ \overline{s(m\Delta)\ s(p\Delta)}\ g(m\Delta)\ g(p\Delta) =$$

$$= \Delta^2 \sum_{mp} h(k_1\Delta-m\Delta)\ h(k_2\Delta-p\Delta)\ R_s(m\Delta-p\Delta)\ g(m\Delta)\ g(p\Delta)\ . \quad (16)$$

When we take explicit account of realizability condition (14) and the finite duration of unity gating function $g(k\Delta)$ in (4), the filter output signal covariance in (16) can be efficiently computed according to

$$C_s(k_1\Delta,k_2\Delta) = \Delta^2 \sum_{m=k_a}^{K_1} h(k_1\Delta-m\Delta) \sum_{p=k_a}^{K_2} h(k_2\Delta-p\Delta)\ R_s(m\Delta-p\Delta)\ , \quad (17)$$

where

$$K_1 = \min(k_1,k_b)\ , \quad K_2 = \min(k_2,k_b)\ . \quad (18)$$

Since $k_b \geq k_a$, $k_1 \geq k_a$, $k_2 \geq k_a$, it follows that $K_1 \geq k_a$ and $K_2 \geq k_a$, thereby making the summations in (17) treat only nonzero entries.

SUMMER CONSIDERATIONS

The weighted summer in figure 1 and (6) - (7) takes its samples at times $k\Delta$, where $k_c \le k \le k_d$. If $k_d < k_a$, then no signal gets into the summer, and the signal at the input cannot be detected at all, whether present or not. Since this is not a useful application of the detector in figure 1, the only situation we will address is that where $k_d \ge k_a$, so that some signal (when present) contributes to the summer output.

There are then two possibilities for observation start time $k_c\Delta$, as indicated in figure 4 below. If $k_c < k_a$ (case 1), the summer is taking in noise-only samples for $k_c \le k < k_a$; this will degrade performance of the detector but is sometimes unavoidable when the signal onset time, $k_a\Delta$, is unknown. In this case, we only need to compute signal output covariance $C_s(k_1\Delta,k_2\Delta)$ for $k_a \le k_1,k_2 \le k_d$, since $C_s = 0$ for $k_1 < k_a$ or $k_2 < k_a$.

On the other hand, if $k_c \ge k_a$ (case 2), signal is taken into the summer on all samples available to it. However, if k_c is considerably larger than k_a, a significant portion of the signal contribution can be lost; this degradation of performance is

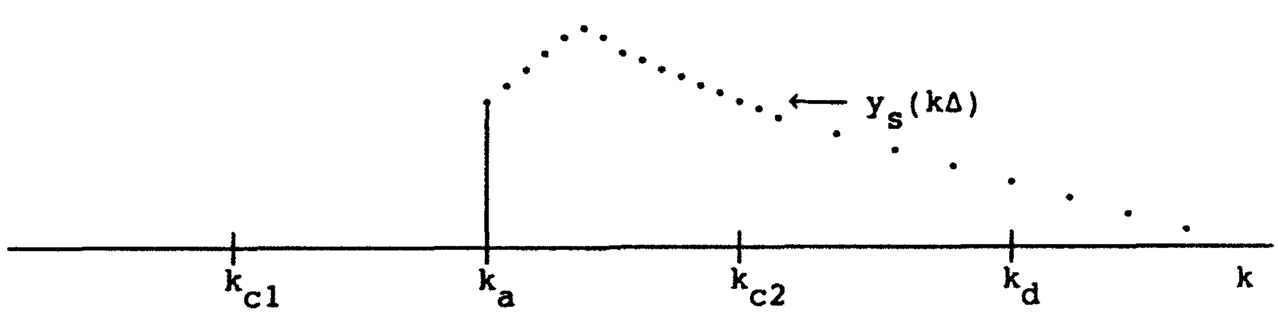

Figure 4. Filter Output Signal $y_s(k\Delta)$

sometimes unavoidable when k_a is unknown. In this latter case, we only need to compute $C_s(k_1\Delta,k_2\Delta)$ for $k_c \le k_1,k_2 \le k_d$.

The general rule is that we need to compute, from (17), the filter output signal covariance $C_s(k_1\Delta,k_2\Delta)$ for

$$K_o \equiv \max(k_a,k_c) \le k_1,k_2 \le k_d \; . \tag{19}$$

Then, if $k_c < k_a$, use 0 for $C_s(k_1\Delta,k_2\Delta)$ for those values where $k_1 < k_a$ or $k_2 < k_a$. Of course, noise covariance $C_n(k_1\Delta,k_2\Delta)$ in (9) or (12) must be computed for $k_c \le k_1,k_2 \le k_d$ in all cases.

Since, from (19),

$$k_1 \ge \max(k_a,k_c) \ge k_a, \quad \text{then} \quad K_1 = \min(k_1,k_b) \ge k_a \; ; \tag{20}$$

also, since

$$k_2 \ge \max(k_a,k_c) \ge k_a, \quad \text{then} \quad K_2 = \min(k_2,k_b) \ge k_a \; . \tag{21}$$

Therefore, the sums in (17) always have some entries; that is, the upper limit is never less than the lower limit. The only restriction is

$$k_a \le k_d \; . \tag{22}$$

Of course, we must always have $k_a \le k_b$ and $k_c \le k_d$.

In summary, the filter output signal covariance follows from (17) and (18), while the filter output noise covariance is available in (9) or (12). The region where the filter output signal covariance $C_s(k_1\Delta,k_2\Delta)$ calculation is nonzero is crosshatched in figure 5 for the case where $k_c < k_a$. If $k_c \ge k_a$, then $C_s(k_1\Delta,k_2\Delta)$ is nonzero in the larger square indicated in figure 5.

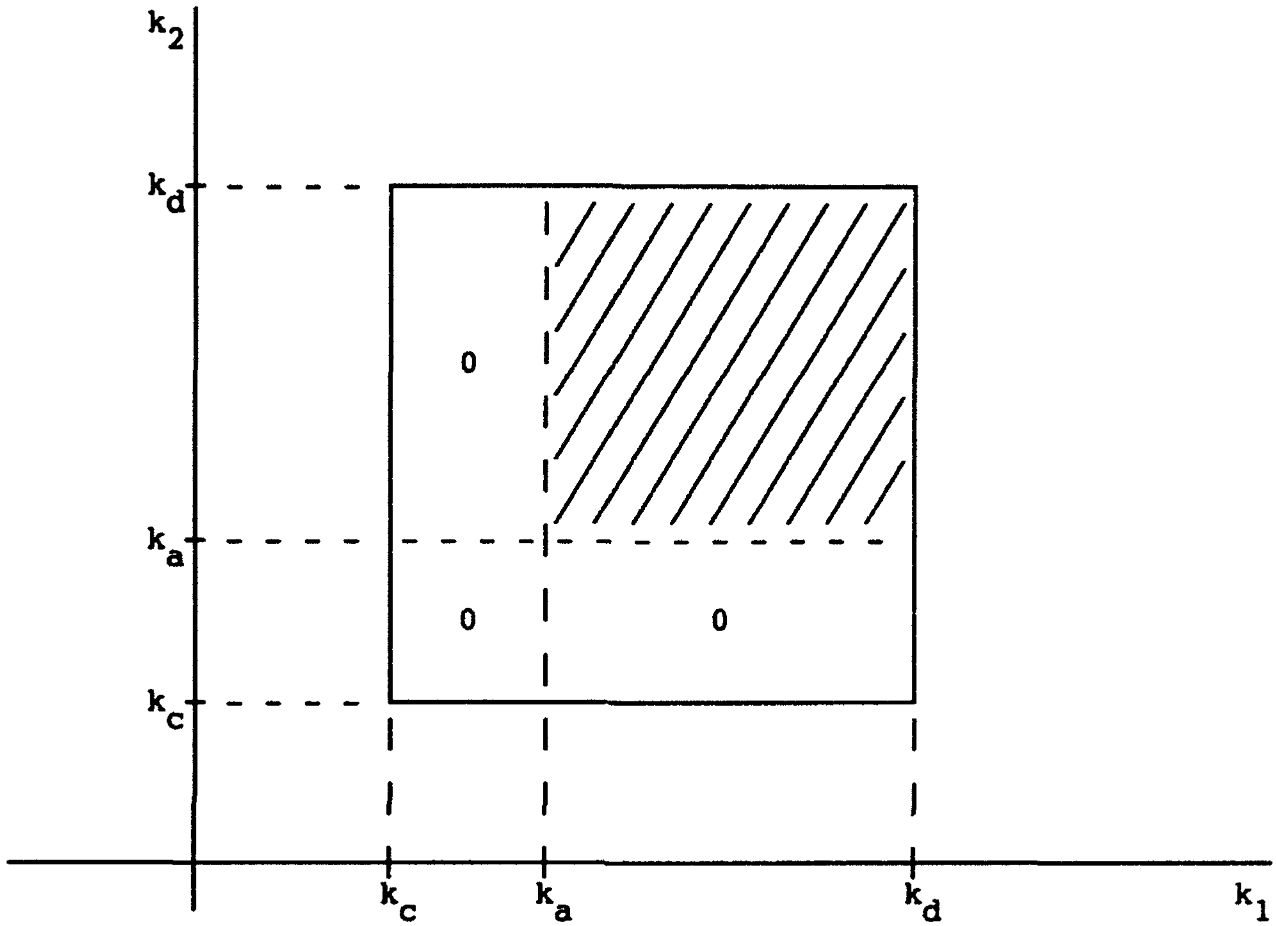

Figure 5. Calculation Region for $C_s(k_1\Delta,k_2\Delta)$

CHARACTERISTIC FUNCTION OF SYSTEM OUTPUT z

The system output z is given in (6) as a sum of weighted and squared correlated zero-mean Gaussian random variables $y(k\Delta)$. Also, the covariance of the noise component $y_n(k\Delta)$ is given by (9) or (12), while the covariance of the signal component $y_s(k\Delta)$ (if present) is given by (17) - (18) and figure 5. Therefore, the covariance of $y(k\Delta)$ is given by

$$C_y(k_1\Delta,k_2\Delta) = C_s(k_1\Delta,k_2\Delta) + C_n(k_1\Delta,k_2\Delta) \; . \qquad (23)$$

The system input signal-to-noise ratio is $R_s(0)/R_n(0)$ in terms of input covariances (2) and (3).

For nonnegative weights $\{w(k\Delta)\}$, we define random variables

$$a_k = \sqrt{w(k\Delta)}\; y(k\Delta) \quad \text{for } k_c \le k \le k_d \; . \qquad (24)$$

Then, (6) - (7) yields the output in the form

$$z = \sum_{k=k_c}^{k_d} a_k^2 \; , \qquad (25)$$

where zero-mean Gaussian random variables $\{a_k\}$ have covariance

$$C_a(k_1,k_2) = \overline{a_{k_1} a_{k_2}} = \sqrt{w(k_1\Delta)\; w(k_2\Delta)}\; C_y(k_1\Delta,k_2\Delta) \; . \qquad (26)$$

In order to find the characteristic function of random variable z in (25), we consider the square symmetric covariance matrix C with elements (26) for $k_c \le k_1,k_2 \le k_d$. Let Q be the

normalized modal matrix of the orthonormal eigenvectors of C and let Λ be the diagonal matrix of the eigenvalues $\{\lambda_k\}$ of C; see [1; pages 36 - 39]. Then we have

$$Q^T Q = Q Q^T = I , \quad Q^T C Q = \Lambda . \tag{27}$$

Now let column vector A be made up of elements $\{a_k\}$ for $k_c \le k \le k_d$, as defined in (24), in which case (25) and (26) can be expressed as

$$z = A^T A , \quad C = \overline{A A^T}. \tag{28}$$

Also, let column vector B be defined by linear transformation

$$B = Q^T A ; \text{ then } A = Q B . \tag{29}$$

Substitution of (29) into (28) yields

$$z = A^T A = B^T Q^T Q B = B^T B = \sum_{k=k_c}^{k_d} b_k^2 , \tag{30}$$

where $\{b_k\}$ are the elements of B. At the same time, the covariance matrix of vector B is

$$\overline{B B^T} = Q^T \overline{A A^T} Q = Q^T C Q = \Lambda , \tag{31}$$

where we used (29), (28), and (27). Thus, the sum for z in (30) is composed of uncorrelated (and therefore independent) zero-mean Gaussian random variables $\{b_k\}$, where the variance of element b_k is eigenvalue λ_k.

The characteristic function of random variable z in (30) is,

using the independence and Gaussian character of $\{b_k\}$,

$$f_z(\xi) = \overline{\exp(i\xi z)} = \prod_{k=k_c}^{k_d} \overline{\exp\left(i\xi b_k^2\right)} =$$

$$= \prod_{k=k_c}^{k_d} \int du \ \exp(i\xi u^2) \ (2\pi\lambda_k)^{-\frac{1}{2}} \exp\left(\frac{-u^2}{2\lambda_k}\right) =$$

$$= \prod_{k=k_c}^{k_d} \left(1 - i2\xi\lambda_k\right)^{-\frac{1}{2}} . \qquad (32)$$

Although this final result is given as a finite product of k_d-k_c+1 principal-value square roots, it can be computed as a single principal-value square root of a finite product of complex first-order polynomials, provided that the location of the product in the complex plane is tracked from $\xi = 0$; for example, see [2; pages B-1 and B-2].

COVARIANCE AND FILTER EXAMPLES

INPUT SIGNAL CHARACTERIZATION

The input signal covariance will be taken to be a sum of damped exponentials:

$$R_s(\tau) = \sum_{j=1}^{J} \alpha_j \exp(-|\tau|/\tau_{sj}) \quad \text{for all } \tau \ , \tag{33}$$

where the J exponential parameters $\{\tau_{sj}\}$ are all distinct. The scalings $\{\alpha_j\}$ and time constants $\{\tau_{sj}\}$ can be complex, provided that they occur in complex conjugate pairs; that is, input signal covariance $R_s(\tau)$ must be real for all τ. The origin value,

$$R_s(0) = \sum_{j=1}^{J} \alpha_j = \sigma_s^2 \ , \tag{34}$$

is the input power of stationary signal sequence $s(k\Delta)$ in (1) and (3), prior to gating. The input signal spectrum corresponding to (33) is, with $\omega = 2\pi f$,

$$G_s(f) = \int d\tau \ \exp(-i2\pi f\tau) \ R_s(\tau) = \sum_{j=1}^{J} \frac{2 \ \alpha_j \ \tau_{sj}}{1 + (\omega \ \tau_{sj})^2} \ , \tag{35}$$

which is seen to contain 2J distinct poles in the complex f-plane. $G_s(f)$ can contain up to 2J-2 zeros.

For time sampling increment Δ, the sampled input signal covariance is, from (33),

$$R_s(k\Delta) = \sum_{j=1}^{J} \alpha_j \exp(-a_j|k|) \ ; \quad a_j = \frac{\Delta}{\tau_{sj}} \ . \tag{36}$$

The J exponential parameters $\{a_j\}$ are distinct, but they can be complex; in fact, $\{\alpha_j\}$ and $\{a_j\}$ can both be complex provided that they occur in complex conjugate pairs such that $R_s(k\Delta)$ is real for all k. In fact, some of the scalings $\{\alpha_j\}$ can be negative, provided that total spectrum $G_s(f)$ in (35) is nonnegative for all f. The exponential parameters $\{a_j\}$ in (36) must also satisfy Re $a_j > 0$ in order that the covariance tend to zero at $k = \pm\infty$.

For a particular pair of complex conjugate terms, say

$$\alpha_1 = \alpha_o \ , \quad \alpha_2 = \alpha_o^* \ , \quad \tau_{s1} = \tau_o \ , \quad \tau_{s2} = \tau_o^* \ , \quad a_1 = a_o \ , \quad a_2 = a_o^* , \tag{37}$$

the corresponding part of the (continuous) input signal covariance (33) can be expressed in the alternative form

$$R_s^o(\tau) = \alpha_o \exp(-|\tau|/\tau_o) + \alpha_o^* \exp(-|\tau|/\tau_o^*) =$$

$$= 2 \exp(-\omega_r|\tau|) \ [\alpha_r \cos(\omega_i|\tau|) + \alpha_i \sin(\omega_i|\tau|)] \ , \tag{38}$$

where we have defined the real and imaginary parts

$$\alpha_o = \alpha_r + i\alpha_i \ , \qquad \frac{1}{\tau_o} = \omega_r + i\omega_i \ . \tag{39}$$

The sampled signal covariance for component (38) is

$$R_s^o(k\Delta) = 2 \exp(-a_r|k|) \ [\alpha_r \cos(a_i|k|) + \alpha_i \sin(a_i|k|)] \ , \tag{40}$$

where $a_r = \omega_r \Delta$ and $a_i = \omega_i \Delta$.

The spectral component associated with component covariance (38) is, by reference to (35) and (39),

$$G_s^o(f) = \frac{2\,\alpha_o\,\tau_o}{1 + (\omega\,\tau_o)^2} + \frac{2\,\alpha_o^*\,\tau_o^*}{1 + (\omega\,\tau_o^*)^2} =$$

$$= 2\ \mathrm{Re}\ \frac{2\,\alpha_o/\tau_o}{\omega^2 + 1/\tau_o^2} = 4\ \mathrm{Re}\ \frac{(\alpha_r + i\alpha_i)(\omega_r + i\omega_i)}{\omega^2 + (\omega_r + i\omega_i)^2} =$$

$$= 4\ \frac{\alpha_r\,\omega_r\left(\omega_r^2 + \omega_i^2 + \omega^2\right) + \alpha_i\,\omega_i\left(\omega_r^2 + \omega_i^2 - \omega^2\right)}{\left(\omega_r^2 + (\omega - \omega_i)^2\right)\left(\omega_r^2 + (\omega + \omega_i)^2\right)} =$$

$$= 2\ \frac{\alpha_r\,\omega_r - \alpha_i(\omega - \omega_i)}{\omega_r^2 + (\omega - \omega_i)^2} + 2\ \frac{\alpha_r\,\omega_r + \alpha_i(\omega + \omega_i)}{\omega_r^2 + (\omega + \omega_i)^2} = \tag{41}$$

$$= \frac{\alpha_i + i\alpha_r}{\omega + \omega_i + i\omega_r} + \frac{\alpha_i - i\alpha_r}{\omega + \omega_i - i\omega_r} + \frac{-\alpha_i + i\alpha_r}{\omega - \omega_i + i\omega_r} + \frac{-\alpha_i - i\alpha_r}{\omega - \omega_i - i\omega_r}\ .$$

If this spectral component $G_s^o(f)$ goes negative for some ranges of f, it must be accompanied by additional terms in summation (35) to keep the total spectrum $G_s(f)$ nonnegative for all frequencies f. Spectral component (41) has four poles in the complex ω-plane at locations $\omega = \omega_i \pm i\omega_r$ and $\omega = -\,\omega_i \pm i\omega_r$.

Even if coefficients $\{\alpha_j\}$ and $\{\tau_{sj}\}$ in (35) are purely real and the poles are distinct, oscillatory behavior of spectrum $G_s(f)$ is still attainable. For example, with

$$J = 3, \qquad \{a_j\} = 1,\ 8,\ 15, \qquad \{\alpha_j\} = 11,\ -48,\ 75, \tag{42}$$

the spectrum (35) is displayed versus $\omega\Delta$ in figure 6. Actually plotted is the scaled spectrum

$$\frac{G_s(f)}{2\Delta} = \sum_{j=1}^{J} \frac{\alpha_j\, a_j}{(\omega\Delta)^2 + a_j^2} \,, \tag{43}$$

where we have used (36) to eliminate $\{\tau_{sj}\}$. The negative coefficient for α_2 causes the dip near $\omega\Delta = 4$.

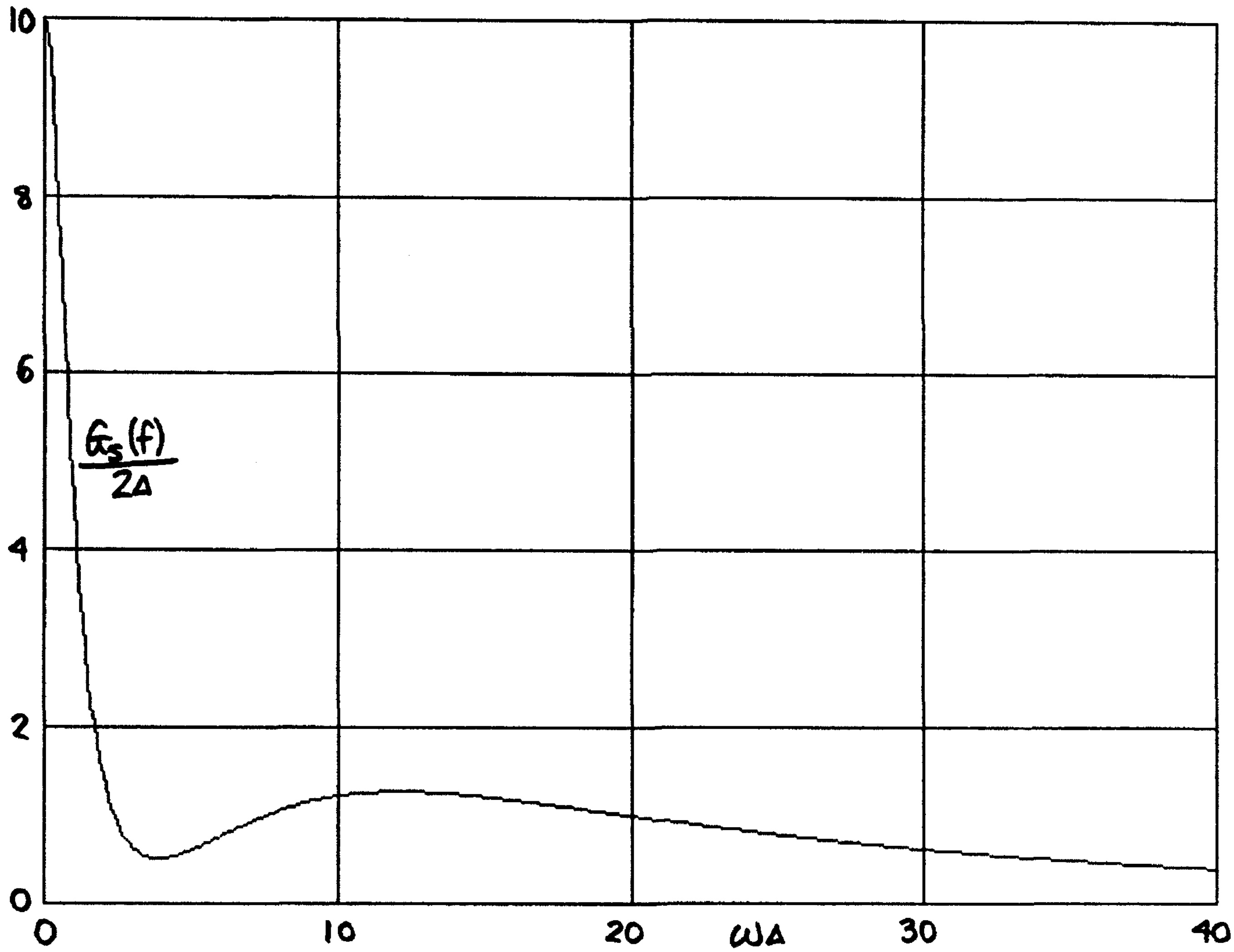

Figure 6. Spectrum (43) for Parameters (42)

INPUT NOISE CHARACTERIZATION

The presentation in this subsection exactly parallels that above for the input signal. The input noise covariance is again taken to be a sum of damped exponentials of the form

$$R_n(\tau) = \sum_{m=1}^{M} \beta_m \exp(-|\tau|/\tau_{nm}) \quad \text{for all } \tau \ , \tag{44}$$

where the M (complex) noise time constants $\{\tau_{nm}\}$ are all distinct from one another. (Some of them may equal some of the signal time constants $\{\tau_{sj}\}$ in (33), if desired.) The origin value,

$$R_n(0) = \sum_{m=1}^{M} \beta_m = \sigma_n^2 \ , \tag{45}$$

is the input power of noise sequence $n(k\Delta)$ in (1).

For time sampling increment Δ, the sampled input noise covariance is, from (44),

$$R_n(k\Delta) = \sum_{m=1}^{M} \beta_m \exp(-b_m|k|) \ ; \quad b_m = \frac{\Delta}{\tau_{nm}} \ . \tag{46}$$

The noise exponential parameters $\{b_m\}$ are distinct, but they can be complex; however, Re $b_m > 0$. (Some of the $\{b_m\}$ can equal some of the $\{a_j\}$ in (36).) The special case of white noise, $R_n(k\Delta) = \sigma_n^2 \, \delta_{ko}$, is handled by choosing $M = 1$, $\beta_1 = \sigma_n^2$, $\tau_{n1} \to 0$, $b_1 \to +\infty$.

FILTER CHARACTERIZATION

We first consider an analog filter with a rational transfer function with N distinct poles, namely

$$H(f) = \sum_{n=1}^{N} \frac{\psi_n}{1 + i2\pi f\tau_{hn}} , \tag{47}$$

where $\{\tau_{hn}\}$ are distinct (complex) time constants. H(f) can have up to N-1 zeros. The corresponding impulse response is

$$h(\tau) = \begin{Bmatrix} \sum_{n=1}^{N} \frac{\psi_n}{\tau_{hn}} \exp\left(\frac{-\tau}{\tau_{hn}}\right) & \text{for } \tau \geq 0 \\ 0 & \text{for } \tau < 0 \end{Bmatrix} , \tag{48}$$

which is required to be real for all τ. The sampled impulse response is then of the form

$$\Delta\ h(m\Delta) = \sum_{n=1}^{N} \psi_n\ c_n \exp(-c_n m) \quad \text{for } m \geq 0 , \quad c_n = \frac{\Delta}{\tau_{hn}} , \tag{49}$$

where the N filter exponential parameters $\{c_n\}$ are distinct from one another; also, Re $c_n > 0$.

The filter correlation function is available by substitution of (49) in (10), with the result

$$\phi_h(j\Delta) = \sum_{n=1}^{N} \psi_n\ c_n\ \nu_n \exp(-c_n|j|) \quad \text{for all } j , \tag{50}$$

where

$$\nu_n = \sum_{k=1}^{N} \frac{\psi_k\ c_k}{1 - \exp(-c_n - c_k)} \quad \text{for } 1 \leq n \leq N . \tag{51}$$

The form of filter correlation $\phi_h(j\Delta)$ in (50) is identical to the sampled input signal covariance (36), in that (50) is also composed of N distinct exponentials; hence, the comments and examples given there are relevant here also.

A useful alternative exists to the calculation of constants $\{\nu_n\}$ via (51). Namely, define z transform filter

$$\underline{H}(z) = \sum_{m=0}^{\infty} z^{-m} \, \Delta \, h(m\Delta) \ . \tag{52}$$

Replace n by k in (49) and substitute the result into (52); then an interchange of summations yields

$$\underline{H}(z) = \sum_{k=1}^{N} \frac{\psi_k \, c_k}{1 - \frac{1}{z} \exp(-c_k)} \ . \tag{53}$$

It then follows immediately by comparison with (51) that

$$\nu_n = \underline{H}(\exp(c_n)) \quad \text{for } 1 \le n \le N \ . \tag{54}$$

When $\underline{H}(z)$ is available as a rational function, (54) can be used directly, instead of summation (51).

The exponential parameters $\{c_n\}$, in impulse response (49) and corresponding N-pole filter (53), can be complex, provided that they occur in complex conjugate pairs and that their corresponding coefficients $\{\psi_n\}$ are also conjugate pairs. That is, the sum of two conjugate terms of the form of (49) must be real for all $m \ge 0$. This generality allows for oscillatory impulse responses such as yielded by narrowband filters.

FILTER OUTPUT NOISE COVARIANCE

In the sequel, we will find use for the following closed form for the discrete convolution of two exponentials:

$$\exp[-c|k|] \circledast \exp[-b|k|] \equiv \sum_{j} \exp[-c|j|-b|k-j|] =$$

$$= \left\{ \begin{array}{ll} \dfrac{\sinh(c)\exp(-b|k|) - \sinh(b)\exp(-c|k|)}{\cosh(c) - \cosh(b)} & \text{for } b \neq c \\ \\ \left[|k| + \dfrac{\cosh(c)}{\sinh(c)}\right]\exp(-c|k|) & \text{for } b = c \end{array} \right\} . \quad (55)$$

The filter output noise covariance is given by (9). If we define

$$C(k\Delta) = \sum_{j} \phi_h(j\Delta)\, R_n(k\Delta - j\Delta) , \quad (56)$$

then the stationarity of filter output noise $y_n(k\Delta)$ in (8) allows us to express (9) as

$$C_n(k_1\Delta, k_2\Delta) = C(k_1\Delta - k_2\Delta) . \quad (57)$$

Therefore, we can concentrate on evaluation of $C(k\Delta)$ in (56) for representative filter correlations ϕ_h and input noise covariances R_n.

In particular, we will use the general filter correlation (50) and input noise covariance (46), where we presume that none of the filter parameters $\{c_n\}$ are equal to any of the input noise parameters $\{b_m\}$. That is,

$$c_n \neq b_m \quad \text{for all } n,m \, . \tag{58}$$

Substitution of (50) and (46) in (56) and use of (55) results in

$$\begin{aligned} C(k\Delta) &= \sum_j \sum_{n=1}^{N} \psi_n \, c_n \, \nu_n \exp(-c_n|j|) \sum_{m=1}^{M} \beta_m \exp(-b_m|k-j|) = \\ &= \sum_{n=1}^{N} \sum_{m=1}^{M} \psi_n \, c_n \, \nu_n \, \beta_m \sum_j \exp[-c_n|j|-b_m|k-j|] = \\ &= \sum_{m=1}^{M} \mu_m \exp(-b_m|k|) + \sum_{n=1}^{N} \gamma_n \exp(-c_n|k|) \, , \end{aligned} \tag{59}$$

where we defined auxiliary constants

$$\begin{aligned} \mu_m &= \beta_m \sum_{n=1}^{N} \frac{\psi_n \, c_n \, \nu_n \sinh(c_n)}{\cosh(c_n) - \cosh(b_m)} \quad \text{for } 1 \le m \le M \, , \\ \gamma_n &= \psi_n \, c_n \, \nu_n \sum_{m=1}^{M} \frac{\beta_m \sinh(b_m)}{\cosh(b_m) - \cosh(c_n)} \quad \text{for } 1 \le n \le N \, . \end{aligned} \tag{60}$$

Condition (58) keeps all the denominators in (60) from becoming zero. On the other hand, if one of the filter parameters $\{c_n\}$ is equal to one of the noise parameters $\{b_m\}$, then it is merely necessary to utilize instead the second line of (55) for that particular n,m pair in the j sum in the middle line of (59).

End result (59) for the filter output noise covariance is a compact expression that is capable of being quickly computed, once constants $\{\mu_m\}$ and $\{\gamma_n\}$ have been evaluated by means of (60) and stored. Again, we encounter a sum of decaying exponentials, albeit of M + N terms now.

FILTER OUTPUT NOISE COVARIANCE FOR WHITE NOISE INPUT

When the input noise is white, then as noted under (46), we have for the input noise covariance,

$$R_n(k\Delta) = \sigma_n^2 \, \delta_{ko} \, , \quad \text{that is,} \quad M = 1 \, , \; \beta_1 = \sigma_n^2 \, , \; b_1 \to +\infty \, . \tag{61}$$

Use of this result in (60) yields $\mu_1 \to 0$, $\gamma_n \to \sigma_n^2 \, \psi_n \, c_n \, \nu_n$, in which case the filter output noise covariance (59) becomes

$$C(k\Delta) = \sigma_n^2 \sum_{n=1}^{N} \psi_n \, c_n \, \nu_n \exp(-c_n|k|) \, , \tag{62}$$

where $\{\nu_n\}$ are given by (51).

Actually, (62) is a special case of the general white noise result obtained by substitution of (61) into (56); namely, for arbitrary filter $\Delta \, h(m\Delta)$, the filter output noise covariance is

$$C(k\Delta) = \sigma_n^2 \, \phi_h(k\Delta) \quad \text{for white noise input} \, , \tag{63}$$

where filter correlation ϕ_h is given by (10).

FILTER OUTPUT SIGNAL COVARIANCE

The nonstationary filter output signal covariance is given by double summation (17) in conjunction with (18). If we substitute input signal covariance $R_s(k\Delta)$ given by (36), along with filter impulse response $\Delta\ h(m\Delta)$ given by (49), into (17), there follows

$$C_s(k_1\Delta, k_2\Delta) = \sum_{m=k_a}^{K_1} \sum_{n=1}^{N} \psi_n\ c_n\ \exp\left(-c_n(k_1-m)\right) \sum_{p=k_a}^{K_2} \sum_{q=1}^{N} \psi_q\ c_q\ \times$$

$$\times \exp\left(-c_q(k_2-p)\right) \sum_{j=1}^{J} \alpha_j\ \exp\left(-a_j|m-p|\right) =$$

$$= \sum_{j=1}^{J} \alpha_j \sum_{n=1}^{N} \psi_n\ c_n\ \exp\left(-c_n k_1\right) \sum_{q=1}^{N} \psi_q\ c_q\ \exp\left(-c_q k_2\right) \times$$

$$\times \sum_{m=k_a}^{K_1} \exp\left(c_n m\right) \sum_{p=k_a}^{K_2} \exp\left(c_q p - a_j|m-p|\right) . \qquad (64)$$

In order to evaluate the innermost double summation in (64), we let $A = \exp(a_j)$, $B = \exp(c_q)$, $C = \exp(c_n)$ and we presume that $k_1 \le k_2$. It then follows from (18) that $K_1 \le K_2$, and the double sum in the last line of (64) can be developed thusly:

$$S(A,B,C,k_a,K_1,K_2) \equiv \sum_{m=k_a}^{K_1} C^m \sum_{p=k_a}^{K_2} B^p\ A^{-|m-p|} = \qquad (65)$$

$$= \sum_{m=k_a}^{K_1} C^m \left[\sum_{p=k_a}^{m} B^p\ A^{p-m} + \sum_{p=m}^{K_2} B^p\ A^{m-p} - B^m\right] =$$

$$= \sum_{m=k_a}^{K_1} (C/A)^m \sum_{p=k_a}^{m} (BA)^p + \sum_{m=k_a}^{K_1} (CA)^m \sum_{p=m}^{K_2} (B/A)^p - \sum_{m=k_a}^{K_1} (CB)^m \; . \qquad (66)$$

Now we use the fact that

$$\sum_{m=L}^{M} z^m = \frac{z^L - z^{M+1}}{1 - z} \quad \text{for } z \neq 1 \; , \quad L \leq M \; , \qquad (67)$$

and we define auxiliary function

$$g(z,k) \equiv \frac{z^k}{1 - z} \quad \text{for } z \neq 1 \; . \qquad (68)$$

Then, after some amount of manipulation, the sums in (66) can be expressed in the closed form

$$S(A,B,C,k_a,K_1,K_2) = g(CB,k_a) \frac{A^2 - CB}{(A-B)(A-C)} + g(CB,K_1+1) \frac{B(A^2 - 1)}{(A-B)(1-BA)}$$
$$+ g(B/A,K_2+1) \left[g(CA,K_1+1) - g(CA,k_a)\right] - g(BA,k_a) \; g(C/A,K_1+1) \; , \qquad (69)$$

provided that $B \neq A$ and $C \neq A$. (B cannot equal 1/A, nor can C equal 1/A or 1/B, because the real parts of c_n and a_j are always positive.)

We now employ definition (65) in order to express the filter output signal covariance in (64) in the final form

$$C_s(k_1\Delta,k_2\Delta) = \sum_{j=1}^{J} \alpha_j \sum_{n=1}^{N} \psi_n \; c_n \exp\left(-c_n k_1\right) \sum_{q=1}^{N} \psi_q \; c_q \exp\left(-c_q k_2\right) \times$$
$$\times \; S\left(\exp(a_j),\exp(c_q),\exp(c_n),k_a,K_1,K_2\right) \; , \qquad (70)$$

provided that <u>none</u> of the filter parameters $\{c_n\}$ are equal to any of the input signal parameters $\{a_j\}$. That is,

$$c_n \neq a_j \quad \text{for all } n,j \; . \tag{71}$$

This result, (70), cannot be further reduced without the various parameters being specified. The triple summation will not be overly time consuming unless the input signal or filter are characterized by a large number of poles, that is, large J or N.

The result in (70) holds for $k_1 \leq k_2$; the range for $k_1 > k_2$ is most easily handled by use of symmetry relation (15). Also, a significant computational aid is available by using the recurrence relation for the g(z,k) function in (68), namely $g(z,k) = z\, g(z,k-1)$.

In summary, k_a, k_b, k_c, k_d are given integers satisfying $k_a \leq k_b$, $k_c \leq k_d$, $k_a \leq k_d$. Also, $K_o = \max(k_a,k_c)$, $K_1 = \min(k_1,k_b)$, $K_2 = \min(k_2,k_b)$. Since we keep $k_1 \leq k_2$, then $K_1 \leq K_2$. Integers k_1 and k_2 must vary over the range

$$K_o \leq k_1 \leq k_2 \leq k_d \; , \tag{72}$$

meaning that K_1 varies in the range $\min(K_o,k_b)$ to $\min(k_d,k_b)$. From (69), we see that we have to compute function values $g(CB,k_a)$, $g(CA,k_a)$, $g(BA,k_a)$, as well as the arrays of function values of $g(CB,k)$, $g(CA,k)$, $g(B/A,k)$, $g(C/A,k)$ for k in the range $\min(K_o,k_b)+1$ to $\min(k_d,k_b)+1$.

PROGRAMS

A program listing for the white noise input case is presented in appendix D; it consists of two separate parts. The first part computes the filter output signal and noise covariance matrices for unity input signal and noise powers, the total filter output covariance matrix of elements C_y in (23) for the various signal-to-noise ratios of interest, the weighted covariance matrix C of elements C_a in (26) and its corresponding eigenvalue matrix Λ in (27). The output of these eigenvalues to storage completes the first part.

The second part lists the program that takes in these eigenvalues and computes the exceedance distribution functions, for noise-alone as well as for signal present, from characteristic function (32); these are the false alarm and detection probabilities, respectively. The precise method of handling all the widely different signal-to-noise ratios of interest is presented in the next section. At the end of appendix D, there is also a listing of the simulation program utilized to verify these theoretical derivations and results.

The corresponding program for the colored noise case is listed in appendix E. A similar breakdown into two parts has been adopted; however, once the eigenvalues have been computed at the end of the first part, the identical program in part 2 of appendix D is used for distribution calculations and is therefore not listed again. Also, the simulation program use to check the colored noise case is similar to that in part 3 of appendix D.

ON CALCULATION OF RECEIVER OPERATING CHARACTERISTICS

The evaluation of the receiver operating characteristics for a number of different input signal-to-noise ratios typically involves the Fourier transforms of a set of characteristic functions of the decision variable, which frequently have widely different ranges and rates of variation. In order to make these calculations tractable and easily plotted, it is necessary to resort to FFTs [2] with carefully chosen sizes and sampling increments. We now present the reasoning behind our choices of these parameters and their interrelationships.

For ease of presentation, we will consider the numerical evaluation of a probability density function $p_o(u)$ from a given (noise-only) characteristic function $f_o(\xi)$; the extension to an exceedance distribution function [2] is immediate. We have

$$p_o(u) = \frac{1}{2\pi} \int d\xi \, \exp(-i\xi u) \, f_o(\xi) =$$

$$\cong \frac{1}{2\pi} \Delta_o \sum_k \exp(-ik\Delta_o u) \, f_o(k\Delta_o) \equiv \tilde{p}_o(u) \, , \qquad (73)$$

where the trapezoidal approximation with sampling increment Δ_o in ξ has been employed. The approximation $\tilde{p}_o(u)$ defined by (73) is an aliased version of $p_o(u)$ and has period $u_o = 2\pi/\Delta_o$ in u. In order to keep aliasing effects negligible in $\tilde{p}_o(u)$, it will be necessary to choose ξ increment Δ_o small enough that the aliasing lobes at separation u_o don't overlap. Therefore, we can restrict the evaluation of $\tilde{p}_o(u)$ to N_o equally spaced points over that interval u_o, according to

$$\tilde{p}_o\left(\frac{2\pi n}{N_o\Delta_o}\right) = \frac{1}{2\pi}\,\Delta_o \sum_k \exp(-i2\pi kn/N_o)\; f_o(k\Delta_o) \quad \text{for } 0 \le n \le N_o-1 \;. \tag{74}$$

Furthermore, by collapsing the sequence $\{f_o(k\Delta_o)\}$ modulo N_o, (74) can be accomplished exactly and efficiently as an N_o-point FFT when N_o is a power of 2; see [2]. The increment in argument u of $\tilde{p}_o(u)$ in (74) and the maximum useful value of u are

$$\delta_o = \frac{2\pi}{N_o\Delta_o} \;, \qquad u_o = \frac{2\pi}{\Delta_o} \;. \tag{75}$$

Now suppose that we also want to evaluate the probability density function $p_1(u)$ corresponding to a different (signal present) characteristic function $f_1(\xi)$ according to

$$p_1(u) = \frac{1}{2\pi}\int d\xi\, \exp(-i\xi u)\; f_1(\xi) =$$

$$\cong \frac{1}{2\pi}\,\Delta_1 \sum_k \exp(-ik\Delta_1 u)\; f_1(k\Delta_1) \equiv \tilde{p}_1(u) \;, \tag{76}$$

where the sampling increment in ξ is now Δ_1. By identical reasoning to that above, we can get samples of periodic (aliased) approximation $\tilde{p}_1(u)$ according to

$$\tilde{p}_1\left(\frac{2\pi n}{N_1\Delta_1}\right) = \frac{1}{2\pi}\,\Delta_1 \sum_k \exp(-i2\pi kn/N_1)\; f_1(k\Delta_1) \quad \text{for } 0 \le n \le N_1-1 \;, \tag{77}$$

which can be realized as an N_1-point FFT. The increment in u and the maximum useful value of u for this latter case are

$$\delta_1 = \frac{2\pi}{N_1\Delta_1} \;, \qquad u_1 = \frac{2\pi}{\Delta_1} \;. \tag{78}$$

If we now want to eliminate the arguments of functions $\tilde{p}_o(u)$ and $\tilde{p}_1(u)$ and be able to directly plot $\tilde{p}_1(u)$ versus $\tilde{p}_o(u)$ (that is, y versus x), this can be easily accomplished if we force the u increments in (74) and (77) to be equal, namely, $\delta_1 = \delta_o$. But, then, according to (75) and (78), this means choosing

$$N_1 \Delta_1 = N_o \Delta_o \ . \tag{79}$$

Since integers N_o and N_1 will be limited to powers of 2, this will mean repeated halving of the ξ increments Δ_o and Δ_1 in (73) and (76), respectively. The values of $\tilde{p}_1$ yielded by (77) can then be directly plotted versus those values of $\tilde{p}_o$ yielded by (74), up to $n = \min(N_o, N_1) = N_o$.

There are several alternative procedures that could have been adopted. For example, one option is to halve the ξ increment, that is, take $\Delta_1 = \Delta_o/2$, but keep $N_1 = N_o$. This will require discarding every other value put out by FFT (74), in order that the remaining values occur at the same u arguments yielded by (77). This is wasteful of FFT (74) and has not been adopted here. Nor have we employed the possibility of interpolation of (77) to determine approximately what the values of $\tilde{p}_1$ would be at the arguments of $\tilde{p}_o$ in (74).

The need to decrease the size of the ξ increment from Δ_o in (74) to Δ_1 in (77) is fundamental. It arises from the fact that as the input signal-to-noise ratio is increased, the contribution of the corresponding probability density function $p_k(u)$, $k = 0,1,2,\ldots$, moves to higher values on the u scale (thereby

leading to the desired higher detection probabilities). In order that approximation $\tilde{p}_k(u)$ not be severely aliased, the ξ increment Δ_k must therefore be decreased; for example, see upper limits u_o and u_1 in (75) and (78), respectively.

With these points in mind, the following procedure has been adopted and utilized in the programs in this report. First, for the noise-only probability density function $p_o(u)$, a satisfactory value for ξ-increment Δ_o in (73) is found such that aliasing is insignificant in $\tilde{p}_o(u)$. This selection of Δ_o is arrived at by looking at a plot of (74) and modifying Δ_o appropriately by trial and error. An FFT size of $N_o = 128$ is utilized to evaluate (74), which is then stored; this size for N_o has generally proven to be sufficient to keep track of the variations of $p_o(u)$.

When we encounter the characteristic function $f_1(\xi)$ and probability density function $p_1(u)$ for the first (lowest) nonzero signal-to-noise ratio of interest, it is usually necessary to decrease the ξ increment to $\Delta_1 = \Delta_o/2$ or $\Delta_o/4$ in order to control the aliasing inherent in approximation $\tilde{p}_1(u)$. At the same time, N_1 is scaled up by 2 or 4, according to (79), thereby maintaining the same u arguments for (74) and (77), as desired. Again, a plot of aliased version $\tilde{p}_1(u)$, obtained by means of (77), serves as the decision point for making an acceptable choice of value for Δ_1.

For the successively larger signal-to-noise ratios and their corresponding probability density functions $p_k(u)$, $k=1,2,\ldots$, <u>occasional</u> repeated halvings of the ξ increment according to

$\Delta_k = \Delta_{k-1}/2$ are made, but only when needed to keep aliasing insignificant in $\tilde{p}_k(u)$. At those times that such a halving is necessary and made, the FFT size is doubled according to $N_k = 2\ N_{k-1}$. Otherwise, Δ_k and N_k are kept at their same values as used for the k-1 run. By the time the largest signal-to-noise ratio case of interest is reached, the FFT size N_k can get rather large. We are limited in our application to a maximum FFT size of 16384; if aliasing is significant at this size, we are unable to accurately numerically handle these larger signal-to-noise ratios by the adopted procedure. Interpolation would then be the recommended alternative, as alluded to above.

The procedure above has been explained in terms of probability density functions rather than the exceedance distribution functions which are of actual interest. That is, the relevant exceedance distribution functions are plots of the detection probabilities for various signal-to-noise ratios versus the false alarm probability. The only basic change is to replace the top lines of (73) and (76) by the exceedance distributions, which are also obtained by means of Fourier transforms, namely [2; (5) - (6)]

$$E_k(u) = \frac{1}{2} + \int_{0+}^{+\infty} d\xi \ \mathrm{Im}\left\{\exp(-i\xi u)\ \frac{f_k(\xi)}{\pi\xi}\right\} , \tag{80}$$

for k = 0,1,2,...

RESULTS

COMPARISON WITH SIMULATION

In figure 7, the cumulative (CDF) and exceedance distribution functions (EDF) for a white noise input excitation are presented. The input signal and filter are both characterized by one pole, that is, $J = 1$ in (33) - (36) and $N = 1$ in (47) - (50). The simulation employed 1E6 trials and verifies the theoretical calculations down to the probability level of 1E-5 plotted. A listing of the simulation program, including all the parameter values that were utilized here, is given at the end of appendix D. The gating and observation parameters are $k_a = 4$, $k_b = 11$ and $k_c = 2$, $k_d = 16$, respectively; thus, it can be seen that the transient buildup and decay are a prominent part of the filter output signal for this particular example.

The results in figure 8 pertain to a colored noise input with one pole, that is, $M = 1$ in (44) - (46). Again, 1E6 trials were used in the simulation and they confirm the theoretical result down to the 1E-5 level of probabilities. A listing of the simulation program is given at the end of appendix D.

Absolute time is unimportant to the performance of the energy detector in figure 1. That is, any common constant can be added to or subtracted from k_a, k_b, k_c, k_d, without changing the receiver operating characteristics. Thus, k_a could always be selected as 0 without loss of generality.

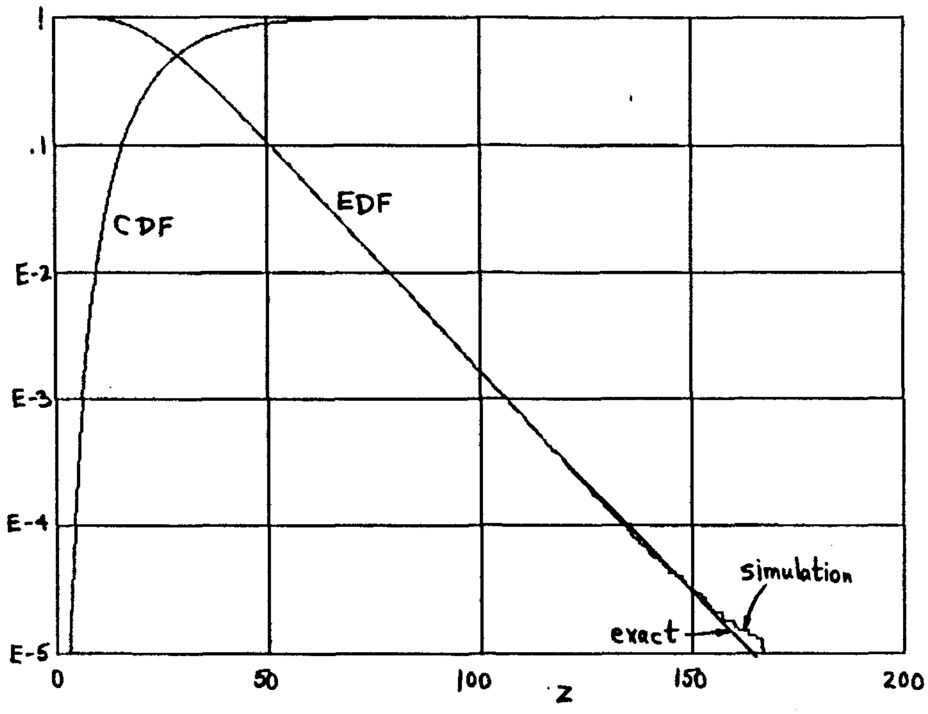

Figure 7. Cumulative and Exceedance Distributions for White Noise

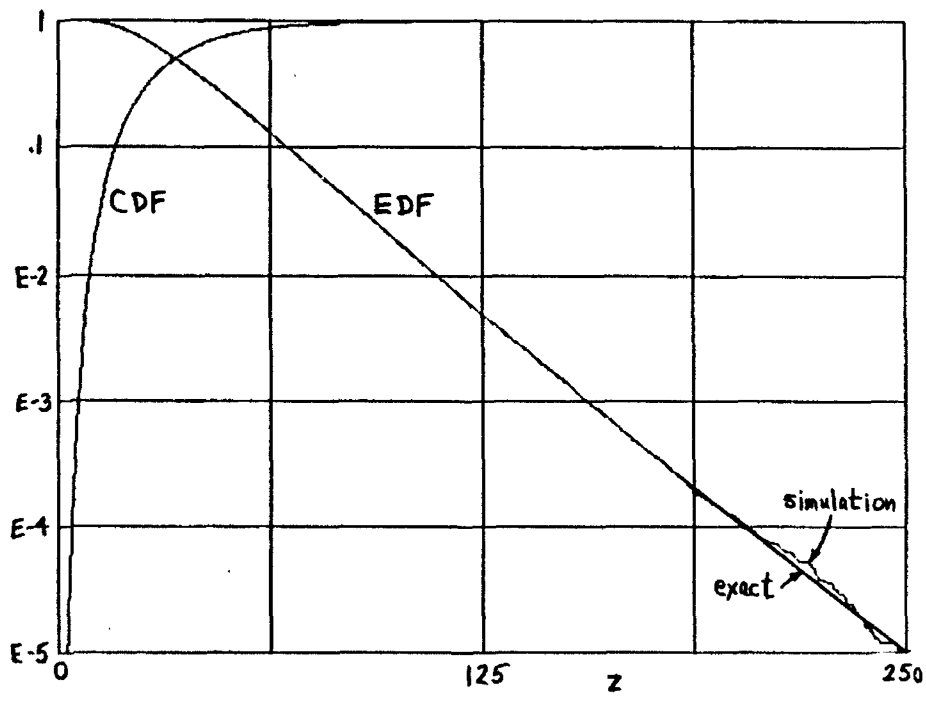

Figure 8. Cumulative & Exceedance Distributions for Colored Noise

RECEIVER OPERATING CHARACTERISTICS

The receiver operating characteristics, that is, detection probability P_D versus false alarm probability P_F for a set of signal-to-noise ratios, for the system in figure 1 will be considered in this section. The first example is evaluated for the following set of parameters, where all times are in seconds. Also, all parameters are real.

$$\begin{aligned} &k_a = 4,\ k_b = 11,\ k_c = 2,\ k_d = 16,\ \Delta = .2, \\ &J = 3,\ \{\alpha_j\} = \{11 \quad -48 \quad 75\},\ \{\tau_{sj}\} = \{1 \quad 1/8 \quad 1/15\}, \\ &M = 2,\ \{\beta_m\} = \{39 \quad 60\},\ \{\tau_{nm}\} = \{.2 \quad .4\}, \\ &N = 4,\ \{\psi_n\} = \{1 \quad 2 \quad 3 \quad 4\},\ \{\tau_{hn}\} = \{.3 \quad .5 \quad .7 \quad .9\}. \end{aligned} \qquad (81)$$

Twenty nonzero values of the system input signal-to-noise ratio

$$R = \frac{\sigma_s^2}{\sigma_n^2} \qquad (82)$$

were utilized, namely R = 5(1.2)27.8 dB. For computation of the exceedance distribution function P_F directly from the characteristic function, the initial ξ increment Δ_o in (73) and (74) was taken as .00025; this yielded round-off errors less than 1E-15 in the false alarm probability calculation. Repeated occasional halving of the ξ increment, as explained in the previous section, was utilized, eventually requiring an FFT size of 16384 for the largest signal-to-noise ratios cases for P_D.

The receiver operating characteristics are plotted on normal probability paper in figure 9 and cover a wide range of values,

ranging from 1E-10 for P_F to .999 for P_D. Since Gaussian random variables would plot on such paper as a set of parallel straight lines, the curvatures of these results illustrate that the Gaussian assumption for decision variable z in figure 1 is not warranted, at least for this example. The major reason for this behavior is the relatively small number of samples used in the detector; see the top line of (81). This is also a partial explanation for the rather large values of the per-sample signal-to-noise ratio R required at the high quality region at the top left of figure 9. Another factor to notice is that the input signal duration is only $k_b\Delta - k_a\Delta = 1.4$ seconds, meaning that the filter output signal never reaches steady state before the input signal is turned off; all these transient signal effects and their limitations on performance have been included exactly in the analysis and numerical results presented here.

The effect of halving the sampling increment Δ in figure 1 is considered in the next example but done in such a manner as to keep the total input signal duration the same. That is, the parameter values in (81) are kept the same except for the following changes:

$$k_a = 8, \; k_b = 22, \; k_c = 4, \; k_d = 32, \; \Delta = .1. \tag{83}$$

Notice that all absolute times, such as $k_a\Delta$, are kept fixed. The corresponding receiver operating characteristics are plotted in figure 10. Performance has been degraded by a couple of dB. For example, to realize $P_F = 1E-3$ and $P_D = .5$, the per-sample input

signal-to-noise ratio R must now be 16 dB, whereas it was 13.8 dB in the previous figure. This is a degradation of 2.2 dB. The main reason for this behavior is again the inability of the filter output signal to reach steady state and thereby contribute significantly to the summer output. By contrast, the filter output noise is in steady state (by assumption) for both examples.

We now return to the original sampling increment $\Delta = .2$ seconds employed in (81). When the observation interval coincides with the input signal nonzero-excitation duration, that is, $k_a = k_c = 4$, $k_b = k_d = 11$, the receiver operating characteristics in figure 11 illustrate additional improvement. For example, probabilities $P_F = 1E-3$ and $P_D = .5$ can now be realized with R = 13.1 dB, which is an improvement of .7 dB relative to the signal-to-noise ratio required for the broader observation interval, $k_c = 2$, $k_d = 16$, used in (81).

Finally, an example with a wide observation interval, namely $k_c = 0$, $k_d = 25$, is displayed in figure 12. The desired probabilities can now be achieved only if the input signal-to-noise ratio is increased to 14.7 dB, a degradation of 1.6 dB relative to the best case above.

These examples illustrate the utility of being able to investigate quantitatively and accurately the effects of nonstationarity in a system, without having to make questionable assumptions about, for example, how closely steady state was or was not realized. They also afford a dependable verification or

rejection of the Gaussian approximation for the decision variable; in a related work [3], the latter approximation was found to be too optimistic for most working ranges of this detection system.

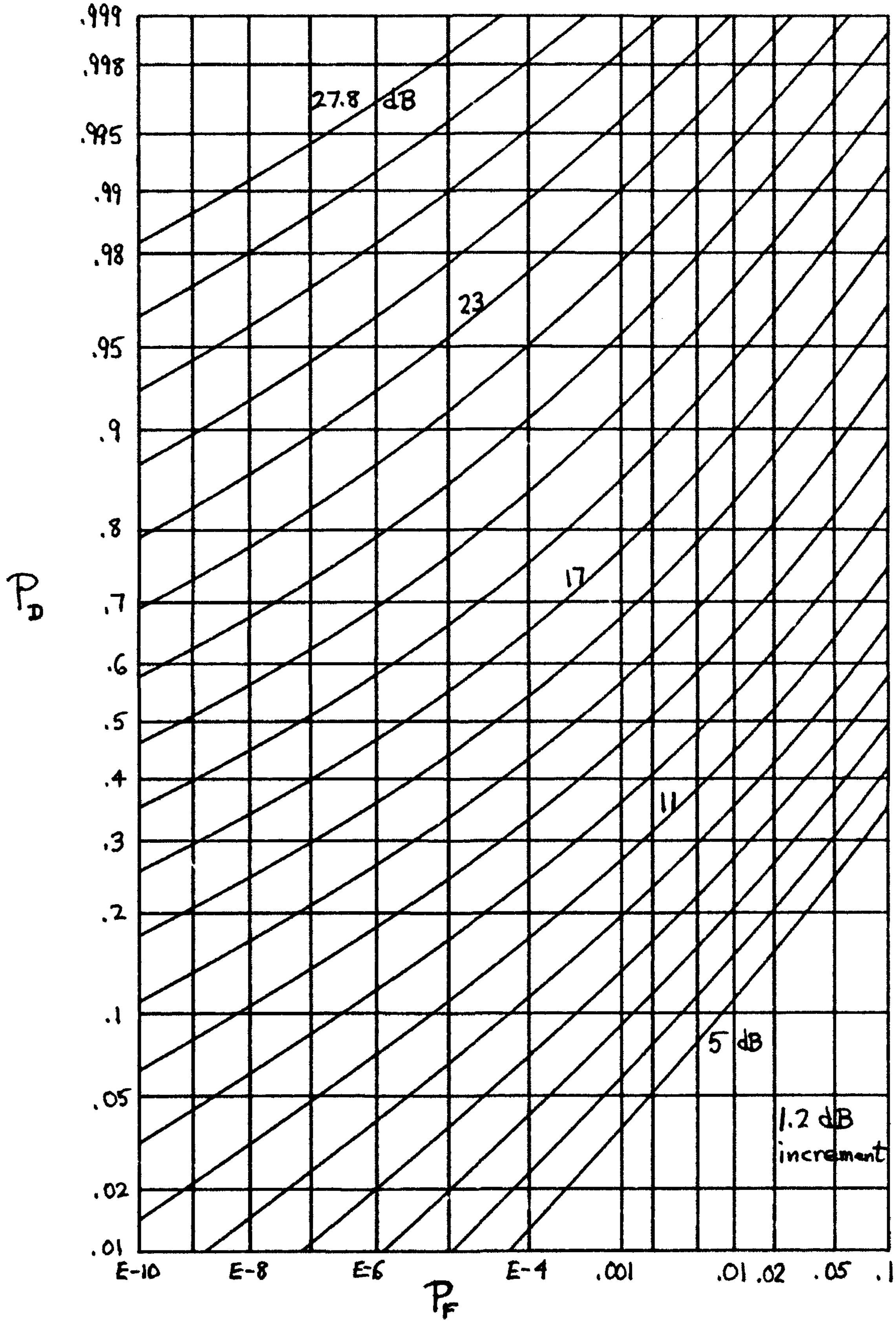

Figure 9. Receiver Operating Characteristics; Example A

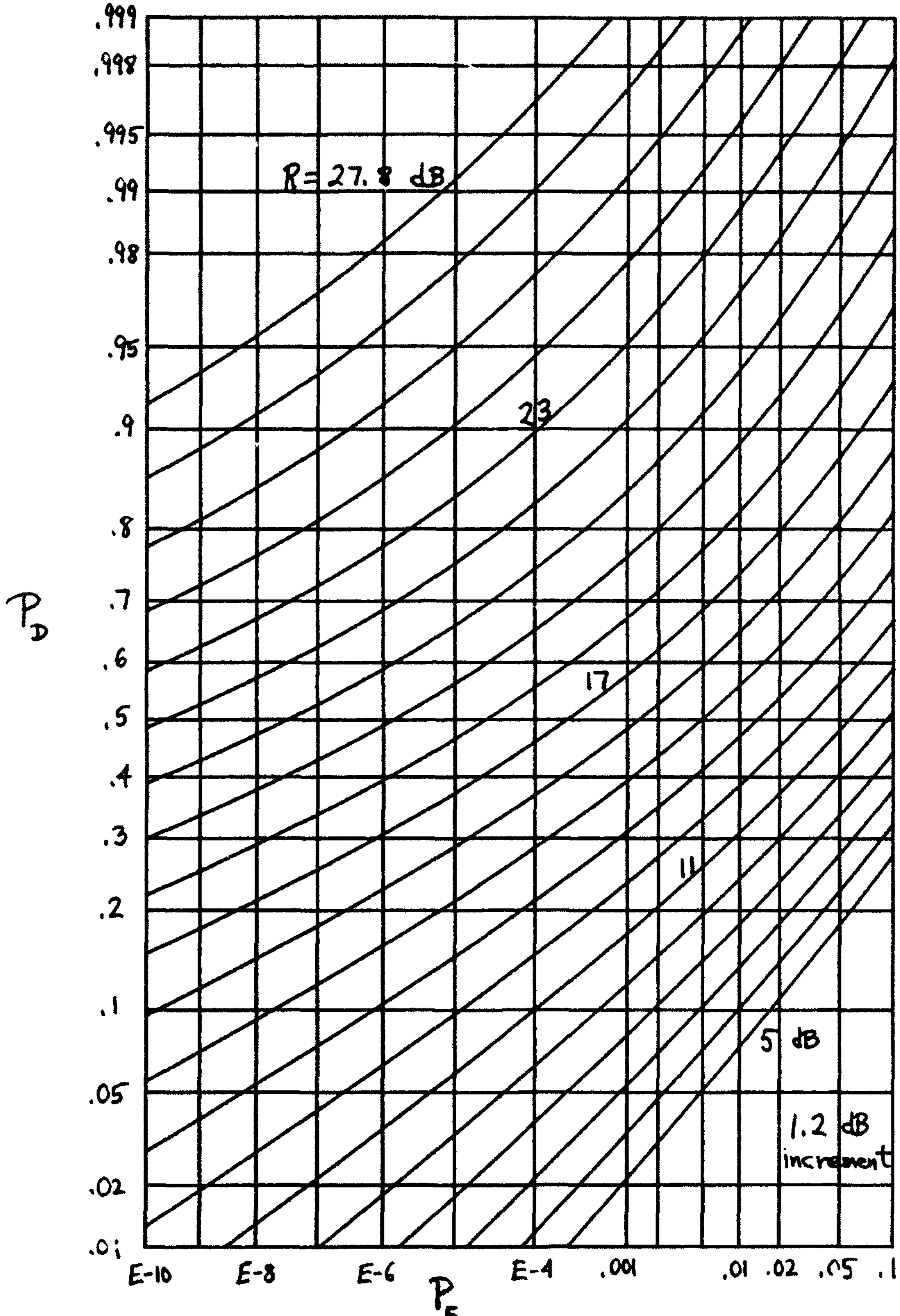

Figure 10. Receiver Operating Characteristics; Example B

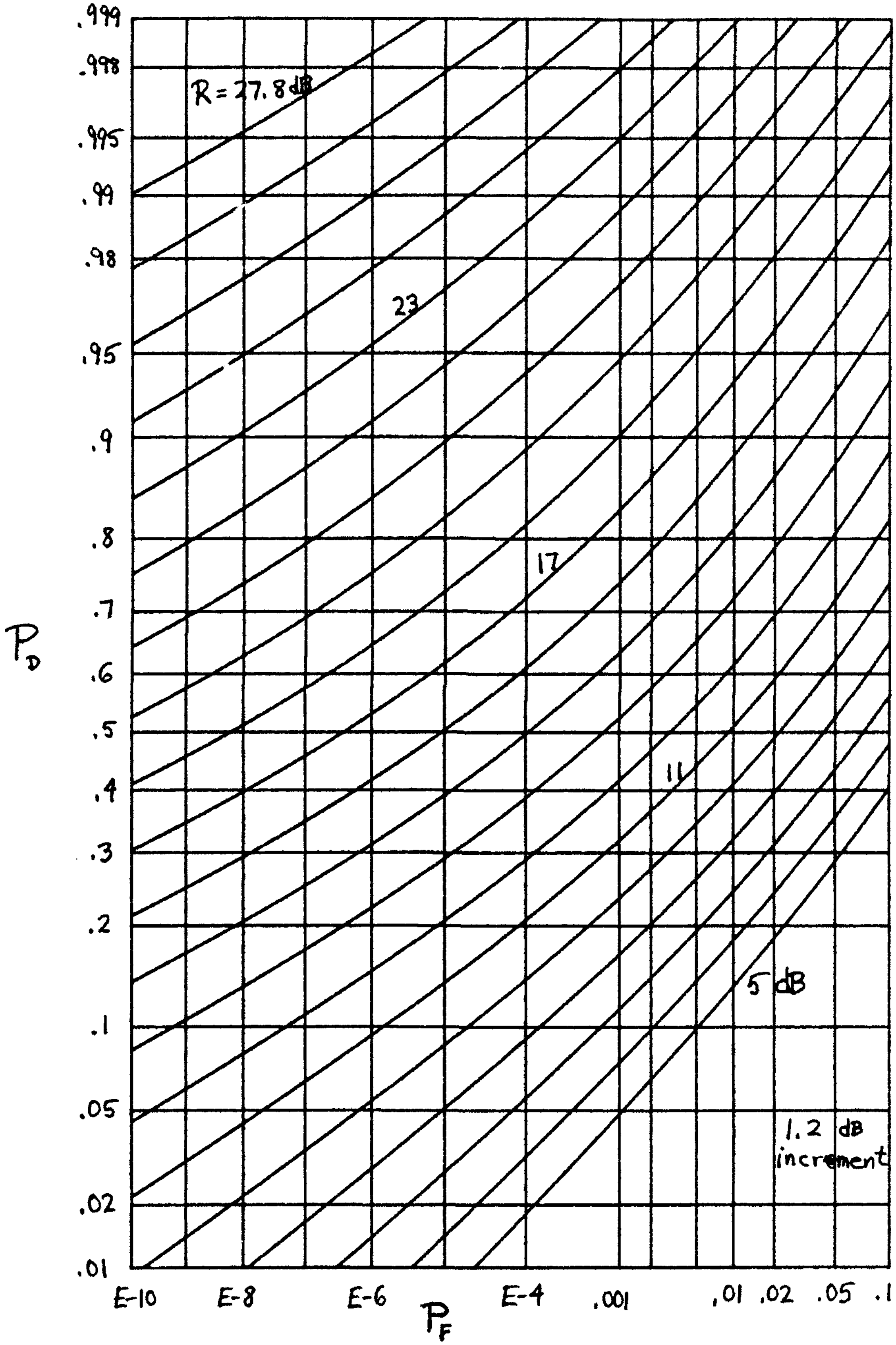

Figure 11. Receiver Operating Characteristics; Example C

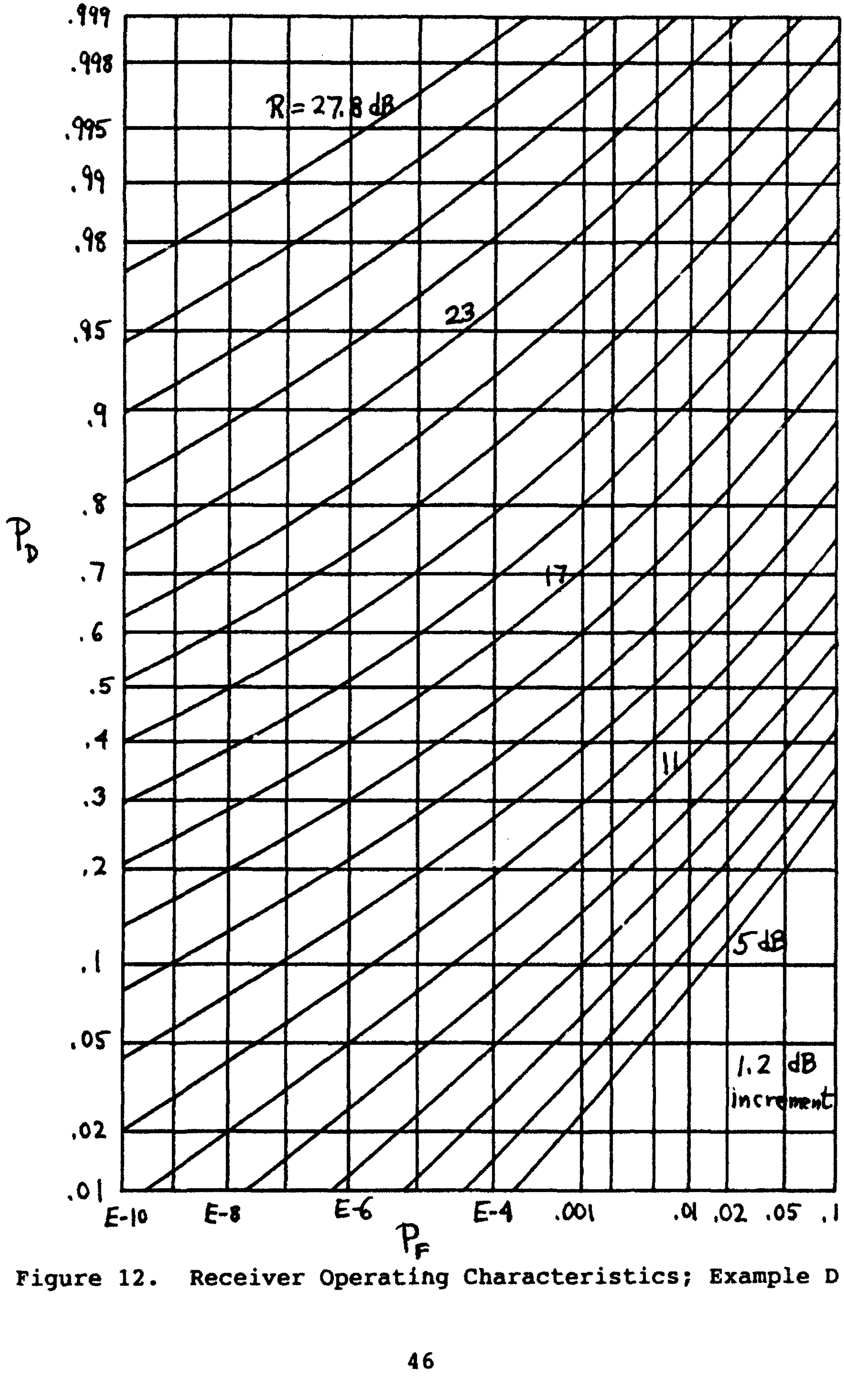

Figure 12. Receiver Operating Characteristics; Example D

SUMMARY

Programs have been written which enable exact analysis of performance of the mismatched bandpass energy detector of figure 1, both for the white noise input case as well as the colored noise input case. These results allow for arbitrary sampling time increment Δ and for arbitrary input signal spectra, input noise spectra, and filter transfer functions. By these means, exact quantitative evaluations and degradations can be determined for various combinations of uncertainty regarding the input signal time location and duration as well as its center frequency, bandwidth, and spectrum. Included in the analysis and programs are the buildup and decay (or any portion thereof) of the nonstationary output signal from the filter, when excited by a burst-like input signal, regardless of the lengths or locations of the excitation and observation intervals. Both programs have been compared with simulation results and confirmed down to as low a probability level as possible consistent with the number of trials used.

APPENDIX A. NUMERICAL EVALUATION OF NOISE COVARIANCE R_n

If the noise covariance $R_n(\tau)$ is not available in analytic form, but the noise spectrum $G_n(f)$ is specified, the following numerical procedure can be employed to evaluate the required samples $R_n(k\Delta)$. We begin with

$$R_n(\tau) = \int df \exp(i2\pi f\tau)\, G_n(f) =$$

$$\cong \Delta_f \sum_m \exp(i2\pi m\Delta_f\tau)\, G_n(m\Delta_f) \equiv \tilde{R}_n(\tau) \;, \qquad \text{(A-1)}$$

where the trapezoidal rule with frequency increment Δ_f was used. But $\tilde{R}_n(\tau)$ in (A-1) can be developed as

$$\tilde{R}_n(\tau) = \int df \exp(i2\pi f\tau)\, G_n(f)\, \Delta_f\, \delta_{\Delta_f}(f) =$$

$$= R_n(\tau) \circledast \delta_{1/\Delta_f}(\tau) = \sum_m R_n\left(\tau - \frac{m}{\Delta_f}\right) . \qquad \text{(A-2)}$$

Now suppose that the noise covariance

$$R_n(\tau) \cong 0 \quad \text{for } |\tau| > \tau_o . \qquad \text{(A-3)}$$

Then, in order to avoid significant aliasing in (A-2), we must take Δ_f small enough that

$$\frac{1}{2\Delta_f} > \tau_o \;, \qquad \text{(A-4)}$$

in which case (A-2) yields

$$\tilde{R}_n(\tau) \cong R_n(\tau) \quad \text{for } |\tau| < \tau_o < \frac{1}{2\Delta_f} . \qquad \text{(A-5)}$$

The required samples of $\tilde{R}_n(\tau)$ follow from (A-1) according to

$$\tilde{R}_n(k\Delta) = \Delta_f \sum_m \exp(i2\pi m\Delta_f k\Delta)\ G_n(m\Delta_f) \ . \tag{A-6}$$

Now take frequency increment

$$\Delta_f = \frac{1}{N\Delta} \ , \tag{A-7}$$

where, according to (A-4), integer N must be taken large enough that $N\Delta/2 > \tau_o$, that is,

$$N > \frac{2\tau_o}{\Delta} \ . \tag{A-8}$$

Then, from (A-5) - (A-7),

$$R_n(k\Delta) \cong \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_m \exp(i2\pi mk/N)\ G_n\left(\frac{m}{N\Delta}\right) \quad \text{for } |k| < \frac{N}{2} \ . \tag{A-9}$$

Alternatively, this can be expressed as

$$R_n(k\Delta) \cong \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_{m=0}^{N-1} \exp(i2\pi mk/N)\ \tilde{G}_n\left(\frac{m}{N\Delta}\right) \quad \text{for } |k| < \frac{N}{2} \ , \tag{A-10}$$

where collapsed (prealiased) noise spectrum and sequence

$$\tilde{G}_n(f) \equiv \sum_j G_n\left(f - \frac{j}{\Delta}\right) \ , \quad \tilde{G}_n\left(\frac{m}{N\Delta}\right) = \sum_j G_n\left(\frac{m}{N\Delta} - \frac{j}{\Delta}\right) \quad \text{for } 0 \le m \le N-1 \ . \tag{A-11}$$

The procedure in (A-10) can be efficiently realized as an N-point FFT of N nonzero numbers. The major conditions that must be met are (A-8) in conjunction with (A-3).

APPENDIX B. NUMERICAL EVALUATION OF FILTER CORRELATION ϕ_h

Suppose that we want digital filter correlation

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta)\, h(m\Delta - j\Delta) \ , \tag{B-1}$$

defined in (10), but all that we have is the filter transfer function H(f), where

$$h(\tau) = \int df \exp(i2\pi f\tau)\, H(f) \quad \text{for all } \tau \ . \tag{B-2}$$

Impulse response $h(\tau)$ is presumed real. The direct relationship between $\phi_h(j\Delta)$ and H(f) is the subject of this appendix.

First, define periodic function

$$\tilde{H}(f) \equiv \sum_m H\left(f - \frac{m}{\Delta}\right) \quad \text{for all } f \ . \tag{B-3}$$

Then, there follows

$$\tilde{H}(f) = H(f) \circledast \delta_{1/\Delta}(f) =$$

$$= \int d\tau \exp(-i2\pi f\tau)\, h(\tau)\, \Delta\, \delta_\Delta(\tau) = \sum_m \exp(-i2\pi f\Delta m)\, \Delta\, h(m\Delta) \ , \tag{B-4}$$

where we used the fact that convolution in the frequency domain is equivalent to Fourier transformation of a product in the time domain. That is, $\tilde{H}(f)$ is the Fourier transform of the samples of impulse response $h(\tau)$ taken at time increment Δ. However, care must be taken at points of discontinuity of $h(\tau)$; for example, if $h(\tau)$ is discontinuous at $k\Delta$, the contribution to the right-hand

side of (B-4), at $m = k$, is $\frac{1}{2}[h(k\Delta+) + h(k\Delta-)]$. This point and an example are discussed more fully in appendix C.

Now, there follows from (B-1) and (B-2),

$$\phi_h(j\Delta) = \Delta^2 \sum_m \int df \exp(i2\pi f\Delta m)\, H(f) \int du \exp[-i2\pi u\Delta(m-j)]\, H^*(u) =$$

$$= \Delta \iint df\, du\, H(f)\, H^*(u) \exp(i2\pi u\Delta j)\, \Delta \sum_m \exp[i2\pi(f-u)\Delta m] =$$

$$= \Delta \iint df\, du\, H(f)\, H^*(u) \exp(i2\pi u\Delta j) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) =$$

$$= \Delta \sum_m \int df\, H(f)\, H^*\left(f - \frac{m}{\Delta}\right) \exp\left[i2\pi\left(f - \frac{m}{\Delta}\right)\Delta j\right] =$$

$$= \Delta \int df \exp(i2\pi f\Delta j)\, H(f) \sum_m H^*\left(f - \frac{m}{\Delta}\right) =$$

$$= \Delta \int df \exp(i2\pi f\Delta j)\, H(f)\, \widetilde{H}^*(f)\ , \tag{B-5}$$

where we used (B-3).

Now, let I_n denote the following interval of length $1/\Delta$ on the f axis:

$$I_n \equiv \left(\frac{n-\frac{1}{2}}{\Delta}, \frac{n+\frac{1}{2}}{\Delta}\right)\ . \tag{B-6}$$

Then, the filter correlation ϕ_h follows from (B-5) according to

$$\phi_h(j\Delta) = \Delta \sum_n \int_{I_n} df \exp(i2\pi f\Delta j)\, H(f)\, \widetilde{H}^*(f) =$$

$$= \Delta \sum_n \int_{I_o} du \, \exp\left[i2\pi\left(u + \frac{n}{\Delta}\right)\Delta j\right] H\left(u + \frac{n}{\Delta}\right) \tilde{H}^*(u) =$$

$$= \Delta \int_{I_o} du \, \exp(i2\pi u\Delta j) \, \tilde{H}^*(u) \sum_n H\left(u + \frac{n}{\Delta}\right) =$$

$$= \Delta \int_{-.5/\Delta}^{.5/\Delta} du \, \exp(i2\pi u\Delta j) \, |\tilde{H}(u)|^2 , \qquad \text{(B-7)}$$

where we let $u = f - n/\Delta$ and used (B-3) and the periodicity of $\tilde{H}(f)$. This result indicates that we must first alias the given transfer function $H(f)$ according to (B-3) and then Fourier transform its magnitude-square over an interval of length $1/\Delta$.

An alternative approach that leads to the same result (B-7) is to use (B-2) and (B-3) in the form

$$h(m\Delta) = \int df \, \exp(i2\pi f\Delta m) \, H(f) = \int_{I_o} df \, \exp(i2\pi f\Delta m) \, \tilde{H}(f) . \qquad \text{(B-8)}$$

Use of this latter result in (B-1) leads to filter correlation

$$\phi_h(j\Delta) = \Delta^2 \sum_m \int_{I_o} df \, \exp(i2\pi f\Delta m) \, \tilde{H}(f) \int_{I_o} du \, \exp[-i2\pi u\Delta(m-j)] \, \tilde{H}^*(u) =$$

$$= \Delta \iint_{I_o} df \, du \, \exp(i2\pi u\Delta j) \, \tilde{H}(f) \, \tilde{H}^*(u) \, \Delta \sum_m \exp[i2\pi(f - u)\Delta m] =$$

$$= \Delta \iint_{I_o} df \, du \, \exp(i2\pi u\Delta j) \, \tilde{H}(f) \, \tilde{H}^*(u) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) =$$

$$= \Delta \int_{I_o} df\ \exp(i2\pi f\Delta j)\ |\tilde{H}(f)|^2 , \tag{B-9}$$

since only one impulse, at $u = f$, lies in interval I_o. Result (B-9) is identical to (B-7). Notice that we do <u>not</u> get

$$\Delta \int df\ \exp(i2\pi f\Delta j)\ |H(f)|^2 = \Delta \int_{I_o} df\ \exp(i2\pi f\Delta j) \sum_n \left|H\left(f + \frac{n}{\Delta}\right)\right|^2 . \tag{B-10}$$

As for the actual numerical evaluation of (B-9), suppose we sample the integral on f with increment $\Delta_f = 1/(K\Delta)$ and use the trapezoidal approximation; then there follows

$$\phi_h(j\Delta) \cong \frac{1}{K} \sum_{k=-K/2}^{K/2} w_k \exp\left(i2\pi \frac{k}{K} j\right) \left|\tilde{H}\left(\frac{k}{K\Delta}\right)\right|^2 , \tag{B-11}$$

where weights $\{w_k\}$ are given by

$$w_k = \begin{Bmatrix} \frac{1}{2} & \text{for } |k| = K/2 \\ 1 & \text{for } |k| < K/2 \end{Bmatrix} . \tag{B-12}$$

By using the periodicity K of the exp and $\tilde{H}$ terms in (B-11), that sum may be written exactly in the standard FFT form

$$\phi_h(j\Delta) \cong \frac{1}{K} \sum_{k=0}^{K-1} \exp(i2\pi jk/K) \left|\tilde{H}\left(\frac{k}{K\Delta}\right)\right|^2 \quad \text{for } |j| < \frac{K}{2} . \tag{B-13}$$

This is a good approximation, provided that the FFT size K satisfies $K > 2\tau_h/\Delta$, where τ_h is the effective duration of filter correlation $\phi_h(\tau)$.

APPENDIX C. DISPLACED SAMPLING AND FOURIER TRANSFORM OF DISCONTINUOUS FUNCTION

GENERAL RELATIONS

Let a(t) and A(f) be a Fourier transform pair:

$$A(f) = \int dt \exp(-i2\pi ft)\, a(t) \; . \qquad \text{(C-1)}$$

Also, let b(t) and B(f) be a Fourier transform pair. Then, Parseval's theorem states that

$$\int dt\, a(t)\, b^*(t) = \int df\, A(f)\, B^*(f) \; . \qquad \text{(C-2)}$$

We now apply this relation to the case where

$$b(t) = \Delta\, \delta_\Delta(t - \alpha) \; , \quad B(f) = \delta_{1/\Delta}(f) \exp(-i2\pi f\alpha) \; . \qquad \text{(C-3)}$$

There follows

$$L \equiv \Delta \sum_n a(n\Delta + \alpha) = \int dt\, a(t)\, \Delta\, \delta_\Delta(t - \alpha) = \qquad \text{(C-4)}$$

$$= \int df\, A(f)\, \delta_{1/\Delta}(f) \exp(i2\pi f\alpha) = \sum_n A\left(\frac{n}{\Delta}\right) \exp(i2\pi n\alpha/\Delta) \equiv R \; . \qquad \text{(C-5)}$$

If the samples required in the sums in (C-4) and (C-5) encounter a discontinuity of a(t) or A(f), the value used in (C-4) or (C-5) must be taken as the average value approached from both sides of the discontinuity. Also, the sums in (C-4) and (C-5) may have to be interpreted as principal value, if necessary for convergence.

EXAMPLE

$$a(t) = \begin{cases} \exp[-(b+ic)t] & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} , \quad b > 0 . \tag{C-6}$$

$$A(f) = \frac{1}{b + i(c + 2\pi f)} \quad \text{for all } f . \tag{C-7}$$

Function $a(t)$ is discontinuous at $t = 0$, while $A(f)$ is continuous for all f.

Let $0 < \alpha < \Delta$; then the sum in (C-4) is

$$L = \Delta \sum_{n=0}^{\infty} \exp[-(b+ic)(n\Delta+\alpha)] = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta . \tag{C-8}$$

At the same time, the sum in (C-5) is

$$R = \sum_{n} \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} \quad \text{for all } \alpha . \tag{C-9}$$

When $\alpha \neq 0$ (or an integer multiple of Δ), the phase factor in the numerator of (C-9) yields a convergent sum for the real and imaginary parts, without the need for a principal value interpretation. It has also been verified numerically that (C-8) and (C-9) are equal, as predicted by (C-4) and (C-5); that is,

$$\sum_{n} \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta . \tag{C-10}$$

On the other hand, if $\alpha = 0$, then the sum in (C-4) is, accounting for the discontinuity of a(t),

$$L = \frac{\Delta}{2} + \Delta \sum_{n=1}^{\infty} \exp[-(b+ic)n\Delta] = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \qquad \text{(C-11)}$$

Also, then, the sum in (C-5) is

$$R = \sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \sum_n \frac{b - i(c + 2\pi n/\Delta)}{b^2 + (c + 2\pi n/\Delta)^2} , \qquad \text{(C-12)}$$

the imaginary part of which must be interpreted as a principal value sum. Again, it has been numerically verified (next page) that (C-11) and (C-12) are equal; that is,

$$\sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \qquad \text{(C-13)}$$

Notice that the limit of (C-10) as $\alpha \to 0+$ does <u>not</u> yield (C-13).

If we apply result (C-11) to $\tilde{H}(f)$ in the last entry in (B-4), we have, for example (304),

$$\tilde{H}(f) = \frac{a}{2} \frac{1 + \exp(-a-i2\pi f\Delta)}{1 - \exp(-a-i2\pi f\Delta)} . \qquad \text{(C-14)}$$

By contrast, the limit of (C-10) as $\alpha \to 0+$ yields

$$\frac{a}{1 - \exp(-a-i2\pi f\Delta)} . \qquad \text{(C-15)}$$

BASIC PROGRAM FOR NUMERICAL VERIFICATION

```
10    Bs=.57                          !  b
20    Cs=.71                          !  c
30    De=.93                          !  delta
40    Al=.1                           !  alpha
50    PRINT Bs,Cs,De,Al
60    CALL Exp(-Al*Bs,-Al*Cs,Nr,Ni)
70    CALL Exp(-De*Bs,-De*Cs,Er,Ei)
80    CALL Div(De*Nr,De*Ni,1-Er,-Ei,Lr,Li)
90    IF Al=0. THEN Lr=Lr-.5*De
100   PRINT Lr,Li                     !  LEFT-HAND SIDE L
110   A=2*PI/De
120   B=A*Al
130   B2=Bs*Bs
140   Rr=Ri=0.
150   DOUBLE Ns                       !  INTEGER
160   FOR Ns=-1E5 TO 1E5              !  PRINCIPAL VALUE SUM
170   T=B*Ns
180   C=COS(T)
190   S=SIN(T)
200   T=Cs+A*Ns
210   D=B2+T*T
220   Dr=(C*Bs+S*T)/D
230   Di=(S*Bs-C*T)/D
240   Rr=Rr+Dr
250   Ri=Ri+Di
260   NEXT Ns
270   PRINT Rr,Ri                     !  RIGHT-HAND SIDE R
280   END
290   !
300   SUB Div(X1,Y1,X2,Y2,U,V)        !  Z1/Z2
310   T=X2*X2+Y2*Y2
320   U=(X1*X2+Y1*Y2)/T
330   V=(Y1*X2-X1*Y2)/T
340   SUBEND
350   !
360   SUB Exp(X,Y,U,V)                !  EXP(Z)
370   E=EXP(X)
380   U=E*COS(Y)
390   V=E*SIN(Y)
400   SUBEND
```

```
.57              .71               .93          0
.729357897734    -.80564044434
.729357647983    -.805640755434

.57              .71               .93          .001
1.19310532814    -.806028668009
1.19353711305    -.806028668087

.57              .71               .93          .1
1.07135515262    -.839119622089
1.07135537888    -.839119622093
```

APPENDIX D. PROGRAMS FOR WHITE INPUT NOISE

Three BASIC programs are listed in this appendix. All the signal, noise, and filter parameters are restricted to be real here. The first program computes the filter output signal and noise covariance matrices for a white noise input, the sums of these matrices scaled by the various signal-to-noise ratios and then weighted, and the eigenvalues which are then stored. The particular subroutine listed here, SUB Svd, actually computes the eigenvectors in addition; a faster alternative would be to replace this subroutine by one which calculates only the eigenvalues. Twenty different nonzero values of the input signal-to-noise ratio are allowed in the program and must be chosen and input by the user.

These eigenvalues serve as the input to the second program which computes the cumulative and exceedance distribution functions according to the method given in [2] and then plots the receiver operating characteristics by elimination of the threshold variable according to the method described in (73) - (80) in the main text.

The third program is the simulation program used to check the two programs above. It is written for single-pole processes and allows for a colored input noise process but could be easily modified to handle more general processes. A run of 1E6 trials took 7 hours on the Hewlett-Packard 9000 computer.

```
10  ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES FOR WHITE
20  ! NOISE INPUT; STORE EIGENVALUES IN "EIG" IN LINE 1720
30    Ka=-10                  !  INPUT SIGNAL START
40    Kb=5                    !  INPUT SIGNAL END; Kb >= Ka
50    Kc=0                    !  ACCUMULATOR  START
60    Kd=5                    !  ACCUMULATOR  END; Kd >= Kc,Ka
70    Delta=.2                !  TIME SAMPLING INCREMENT (SECONDS)
80    J=3                     !  NUMBER OF SIGNAL COMPONENTS
90    DATA 11.,-48.,75.       !  SIGNAL SCALINGS (WE SET SUM = 1)
100   DATA 1.,.125,.066667 !  SIGNAL TIME CONSTANTS (SECONDS)
110   N=4                     !  NUMBER OF FILTER COMPONENTS
120   DATA 1.,2.,3.,4.        !  FILTER SCALINGS (ARBITRARY)
130   DATA .3,.5,.7,.9        !  FILTER TIME CONSTANTS (SECONDS)
140   Nr=20                   !  NUMBER OF SIGNAL-TO-NOISE RATIOS
150   DATA 0,1,2,3,4,5,6,7,8,9,10           !  INPUT SNRS IN DB
160   DATA 11,12,13,14,15,16,17,18,19
170   IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 200
180   PRINT "PROBLEM WITH PARAMETERS"
190   STOP
200   DIM Alpha(10),A(10),Psi(10),C(10),R(20),W(100)
210   DIM Pc(10),Ec(10),Pcv(10),Cn(200),Ea(10),E(0,1000)
220   DIM Gca(100),Gcda(100),Gcb(100),Gbda(100),Cs(5000)
230   DIM Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
240   DOUBLE Ka,Kb,Kc,Kd,J,N,Nr,Ns,Ks,Kdc,Ko  !  INTEGERS
250   DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
260   REDIM Alpha(1:J),A(1:J)
270   READ Alpha(*),A(*)
280   MAT Alpha=Alpha/(SUM(Alpha))
290   MAT A=(Delta)/A
300   REDIM Psi(1:N),C(1:N)
310   READ Psi(*),C(*)
320   MAT C=(Delta)/C
330   REDIM R(1:Nr)
340   READ R(*)
350   REDIM W(Kc:Kd)
360   CALL Weights(Kc,Kd,W(*))
370   MAT W=SQR(W)
380   REDIM Pc(1:N),Ec(1:N) ! FILTER OUTPUT NOISE COVARIANCE
390   FOR Ns=1 TO N
400   C=C(Ns)
410   Pc(Ns)=Psi(Ns)*C
420   Ec(Ns)=EXP(-C)
430   NEXT Ns
440   REDIM Pcv(1:N)
450   FOR Ns=1 TO N
460   E=Ec(Ns)
470   S=0.
480   FOR Ks=1 TO N
490   S=S+Pc(Ks)/(1.-E*Ec(Ks))
500   NEXT Ks
510   Pcv(Ns)=Pc(Ns)*S
520   NEXT Ns
530   Kdc=Kd-Kc
540   REDIM Cn(0:Kdc)
550   MAT Cn=(0.)
560   FOR Ns=1 TO N
570   Pcv=Pcv(Ns)
580   E=Ec(Ns)
```

```
590   P=1.
600   Cn(0)=Cn(0)+Pcv
610   FOR Ks=1 TO Kdc
620   P=P*E
630   Cn(Ks)=Cn(Ks)+Pcv*P
640   NEXT Ks
650   NEXT Ns
660   FOR Js=1 TO J         !  FILTER OUTPUT SIGNAL COVARIANCE
670   A(Js)=EXP(A(Js))
680   NEXT Js
690   Ko=MAX(Ka,Kc)
700   REDIM Ea(1:N),E(1:N,Ko:Kd)
710   FOR Ns=1 TO N
720   E=Ec(Ns)
730   Ea(Ns)=1./E
740   E(Ns,Ko)=EXP(-C(Ns)*Ko)
750   FOR Ks=Ko+1 TO Kd
760   E(Ns,Ks)=E(Ns,Ks-1)*E
770   NEXT Ks
780   NEXT Ns
790   L1=MIN(Ko,Kb)+1
800   L2=MIN(Kd,Kb)+1
810   REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
820   L=Kd-Ko+1
830   REDIM Cs(1:L*(L+1)/2)
840   L11=L1+1
850   FOR Js=1 TO J
860   Al=Alpha(Js)
870   A=A(Js)
880   A1=A*A-1.
890   FOR Ns=1 TO N
900   C=Ea(Ns)
910   Ca=C*A
920   Cda=C/A
930   Ca1=1.-Ca
940   Ac=A-C
950   Ae=Al*Pc(Ns)
960   Gca=Ca^Ka/Ca1
970   Gca(L1)=Ca^L1/Ca1
980   Gcda(L1)=Cda^L1/(1.-Cda)
990   FOR Ks=L11 TO L2
1000  Gca(Ks)=Ca*Gca(Ks-1)
1010  Gcda(Ks)=Cda*Gcda(Ks-1)
1020  NEXT Ks
1030  FOR Qs=1 TO N
1040  B=Ea(Qs)
1050  Aee=Ae*Pc(Qs)
1060  Ba=B*A
1070  Bda=B/A
1080  Cb=C*B
1090  Ba1=1.-Ba
1100  Cb1=1.-Cb
1110  Gba=Ba^Ka/Ba1
1120  Ab=A-B
1130  F=B*A1/(Ab*Ba1)
1140  S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1150  Gcb(L1)=Cb^L1/Cb1
1160  Gbda(L1)=Bda^L1/(1.-Bda)
```

```
1170    FOR Ks=L11 TO L2
1180    Gcb(Ks)=Cb*Gcb(Ks-1)
1190    Gbda(Ks)=Bda*Gbda(Ks-1)
1200    NEXT Ks
1210    Ki=0
1220    FOR Ks1=Ko TO Kd
1230    K1=MIN(Ks1,Kb)+1
1240    S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1250    S3=Gca(K1)-Gca
1260    E=Aee*E(Ns,Ks1)
1270    FOR Ks2=Ks1 TO Kd
1280    Ki=Ki+1
1290    K2=MIN(Ks2,Kb)+1
1300    S=S2+Gbda(K2)*S3        !  SUM S
1310    Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*S
1320    NEXT Ks2
1330    NEXT Ks1
1340    NEXT Qs
1350    NEXT Ns
1360    NEXT Js
1370    L=Kd-Kc+1               !  TOTAL WEIGHTED COVARIANCE MATRIX
1380    REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1390    L11=Kc-1
1400    R=0.
1410    FOR I=0 TO Nr
1420    Ki=0
1430    IF I=0 THEN 1450
1440    R=10.^(R(I)*.1)         !  INPUT POWER SIGNAL-TO-NOISE RATIO
1450    FOR Ks1=Kc TO Kd
1460    W=W(Ks1)                !  WEIGHTS (w(k))
1470    FOR Ks2=Ks1 TO Kd
1480    IF Ks1>=Ko THEN 1510
1490    Cs=0.
1500    GOTO 1530
1510    Ki=Ki+1
1520    Cs=Cs(Ki)
1530    Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
1540    L1=Ks1-L11
1550    L2=Ks2-L11
1560    Cw(L1,L2)=Cw(L2,L1)=Pr
1570    NEXT Ks2
1580    NEXT Ks1
1590    CALL Svd(L,L,Cw(*),Ca(*),D(*))   !  EIGENVALUES
1600    MAT SORT D(*) DES
1610    IF D(L)>0. THEN 1640
1620    PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
1630    PAUSE
1640    PRINT "I =";I;"   CONDITION NUMBER =";D(1)/D(L)
1650    PRINT D(*)
1660    PRINT
1670    FOR Ks=1 TO L
1680    Eig(I,Ks)=D(Ks)         !  STORE EIGENVALUES
1690    NEXT Ks
1700    NEXT I
1710    MASS STORAGE IS ":CS80,7"
1720    ASSIGN #1 TO "EIG"
1730    PRINT #1;Nr,Kc,Kd,Eig(*)
1740    ASSIGN #1 TO *
1750    END
1760    !
```

```
1770    SUB Weights(DOUBLE Kc,Kd,REAL W(*))
1780    DOUBLE Ks              !  INTEGER
1790    F=1.                   !  DECAY FACTOR (DIMENSIONLESS)
1800    W(Kd)=1.
1810    FOR Ks=Kd-1 TO Kc STEP -1
1820    W(Ks)=W(Ks+1)*F        !  EXPONENTIAL WEIGHTS
1830    NEXT Ks
1840    SUBEND
1850    !
1860    SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
1870    ALLOCATE Rv1(1:N)      !  NUMERICAL RECIPES, PAGES 60-64
1880    IF M>=N THEN 1910      !  A(*) IS OVER-WRITTEN
1890    PRINT "M<N IS DISALLOWED"
1900    PAUSE
1910    DOUBLE I,J,K,L,Its,Nm,Jj ! INTEGERS (NOT DOUBLE PRECISION)
1920    G=Scale=Anorm=0.
1930    FOR I=1 TO N
1940    L=I+1
1950    Rv1(I)=Scale*G
1960    G=S=Scale=0.
1970    IF I>M THEN 2250
1980    FOR K=I TO M
1990    Scale=Scale+ABS(A(K,I))
2000    NEXT K
2010    IF Scale=0. THEN 2250
2020    FOR K=I TO M
2030    Aa=A(K,I)=A(K,I)/Scale
2040    S=S+Aa*Aa
2050    NEXT K
2060    F=A(I,I)
2070    G=-SQR(S)
2080    IF F<0. THEN G=-G
2090    H=F*G-S
2100    A(I,I)=F-G
2110    IF I=N THEN 2220
2120    FOR J=L TO N
2130    S=0.
2140    FOR K=I TO M
2150    S=S+A(K,I)*A(K,J)
2160    NEXT K
2170    F=S/H
2180    FOR K=I TO M
2190    A(K,J)=A(K,J)+F*A(K,I)
2200    NEXT K
2210    NEXT J
2220    FOR K=I TO M
2230    A(K,I)=A(K,I)*Scale
2240    NEXT K
2250    W(I)=Scale*G
2260    G=S=Scale=0.
2270    IF (I>M) OR (I=N) THEN 2570
2280    FOR K=L TO N
2290    Scale=Scale+ABS(A(I,K))
2300    NEXT K
2310    IF Scale=0. THEN 2570
```

```
2320    FOR K=L TO N
2330    Aa=A(I,K)=A(I,K)/Scale
2340    S=S+Aa*Aa
2350    NEXT K
2360    F=A(I,L)
2370    G=-SQR(S)
2380    IF F<0. THEN G=-G
2390    H=F*G-S
2400    A(I,L)=F-G
2410    FOR K=L TO N
2420    Rv1(K)=A(I,K)/H
2430    NEXT K
2440    IF I=M THEN 2540
2450    FOR J=L TO M
2460    S=0.
2470    FOR K=L TO N
2480    S=S+A(J,K)*A(I,K)
2490    NEXT K
2500    FOR K=L TO N
2510    A(J,K)=A(J,K)+S*Rv1(K)
2520    NEXT K
2530    NEXT J
2540    FOR K=L TO N
2550    A(I,K)=A(I,K)*Scale
2560    NEXT K
2570    Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
2580    NEXT I
2590    FOR I=N TO 1 STEP -1
2600    IF I>=N THEN 2770
2610    IF G=0. THEN 2740
2620    FOR J=L TO N
2630    V(J,I)=A(I,J)/A(I,L)/G
2640    NEXT J
2650    FOR J=L TO N
2660    S=0.
2670    FOR K=L TO N
2680    S=S+A(I,K)*V(K,J)
2690    NEXT K
2700    FOR K=L TO N
2710    V(K,J)=V(K,J)+S*V(K,I)
2720    NEXT K
2730    NEXT J
2740    FOR J=L TO N
2750    V(I,J)=V(J,I)=0.
2760    NEXT J
2770    V(I,I)=1.
2780    G=Rv1(I)
2790    L=I
2800    NEXT I
2810    FOR I=N TO 1 STEP -1
2820    L=I+1
2830    G=W(I)
2840    IF I>=N THEN 2880
2850    FOR J=L TO N
2860    A(I,J)=0.
2870    NEXT J
```

```
2880    IF G=0. THEN 3050
2890    G=1./G
2900    IF I=N THEN 3010
2910    FOR J=L TO N
2920    S=0.
2930    FOR K=L TO M
2940    S=S+A(K,I)*A(K,J)
2950    NEXT K
2960    F=S/A(I,I)*G
2970    FOR K=I TO M
2980    A(K,J)=A(K,J)+F*A(K,I)
2990    NEXT K
3000    NEXT J
3010    FOR J=I TO M
3020    A(J,I)=A(J,I)*G
3030    NEXT J
3040    GOTO 3080
3050    FOR J=I TO M
3060    A(J,I)=0.
3070    NEXT J
3080    A(I,I)=A(I,I)+1.
3090    NEXT I
3100    FOR K=N TO 1 STEP -1
3110    FOR Its=1 TO 30
3120    FOR L=K TO 1 STEP -1
3130    Nm=L-1
3140    IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3360
3150    IF (ABS(W(Nm))+Anorm)=Anorm THEN 3170
3160    NEXT L
3170    C=0.
3180    S=1.
3190    FOR I=L TO K
3200    F=S*Rv1(I)
3210    Rv1(I)=C*Rv1(I)
3220    IF (ABS(F)+Anorm)=Anorm THEN 3360
3230    G=W(I)
3240    H=SQR(F*F+G*G)
3250    W(I)=H
3260    H=1./H
3270    C=G*H
3280    S=-F*H
3290    FOR J=1 TO M
3300    Y=A(J,Nm)
3310    Z=A(J,I)
3320    A(J,Nm)=Y*C+Z*S
3330    A(J,I)=-Y*S+Z*C
3340    NEXT J
3350    NEXT I
3360    Z=W(K)
3370    IF L<>K THEN 3440
3380    IF Z>=0. THEN 3430
3390    W(K)=-Z
3400    FOR J=1 TO N
3410    V(J,K)=-V(J,K)
3420    NEXT J
3430    GOTO 3970
```

```
3440    IF Its<30 THEN 3470
3450    PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3460    PAUSE
3470    X=W(L)
3480    Nm=K-1
3490    Y=W(Nm)
3500    G=Rv1(Nm)
3510    H=Rv1(K)
3520    F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3530    G=SQR(F*F+1.)
3540    Aa=ABS(G)
3550    IF F<0. THEN Aa=-Aa
3560    F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3570    C=S=1.
3580    FOR J=L TO Nm
3590    I=J+1
3600    G=Rv1(I)
3610    Y=W(I)
3620    H=S*G
3630    G=C*G
3640    Z=SQR(F*F+H*H)
3650    Rv1(J)=Z
3660    C=F/Z
3670    S=H/Z
3680    F=X*C+G*S
3690    G=-X*S+G*C
3700    H=Y*S
3710    Y=Y*C
3720    FOR Jj=1 TO N
3730    X=V(Jj,J)
3740    Z=V(Jj,I)
3750    V(Jj,J)=X*C+Z*S
3760    V(Jj,I)=-X*S+Z*C
3770    NEXT Jj
3780    Z=SQR(F*F+H*H)
3790    W(J)=Z
3800    IF Z=0. THEN 3840
3810    Z=1./Z
3820    C=F*Z
3830    S=H*Z
3840    F=C*G+S*Y
3850    X=-S*G+C*Y
3860    FOR Jj=1 TO M
3870    Y=A(Jj,J)
3880    Z=A(Jj,I)
3890    A(Jj,J)=Y*C+Z*S
3900    A(Jj,I)=-Y*S+Z*C
3910    NEXT Jj
3920    NEXT J
3930    Rv1(L)=0.
3940    Rv1(K)=F
3950    W(K)=X
3960    NEXT Its
3970    NEXT K
3980    SUBEND
```

```
10   ! PLOT ROC FROM EIGENVALUES "EIG", LINES 120 & 1060
20     Delxi=.08                 !  INITIAL INCREMENT ON CHAR. FUNCTION
30     M=128                     !  INITIAL SIZE OF FFT
40     Bs=0.                     !  SHIFT b
50     Tol0=1.E-36               !  TOLERANCE FOR FALSE ALARM PROBABILITY
60     Tol1=1.E-32               !  TOLERANCE FOR DETECTION PROBABILITIES
70     DOUBLE M,Nr,Kc,Kd,Kdc,Ms,I,Ks,Ns,Im,Mm   !  INTEGERS
80     DIM E(0:20,1:100),D(1:100),Cos(0:4096)   !  20 NONZERO SNRs
90     DIM X(0:16383),Y(0:16383),Pr(0:20,0:127)
100    PRINT "Delxi =";Delxi,"M =";M,"b =";Bs
110    MASS STORAGE IS ":CS80,7"
120      ASSIGN #1 TO "EIG"
130    READ #1;Nr,Kc,Kd          !  NO. OF SNRs, START & END OF SUMMER
140    Kdc=Kd-Kc+1
150    REDIM E(0:Nr,1:Kdc),D(1:Kdc)
160    READ #1;E(*)              !  EIGENVALUES
170    GINIT
180    PLOTTER IS "GRAPHICS"
190    GRAPHICS ON
200    REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
210    A=2.*PI/M
220    FOR Ms=0 TO M/4
230    Cos(Ms)=COS(A*Ms)         !  INITIAL QUARTER-COSINE TABLE
240    NEXT Ms
250    Tol=Tol0
260    FOR I=0 TO Nr             !  Nr NONZERO SIGNAL-TO-NOISE RATIOS
270    IF I>0 THEN Tol=Tol1
280    FOR Ks=1 TO Kdc
290    D(Ks)=E(I,Ks)             !  COPY EIGENVALUES
300    NEXT Ks
310    Mux=SUM(D)                !  MEAN OF RANDOM VARIABLE x
320    R=0.                      !  ARGUMENT OF SQUARE ROOT
330    P=1.                      !  POLARITY INDICATOR
340    Muy=Mux+Bs                !  MEAN OF y = x + b
350    MAT X=(0.)
360    MAT Y=(0.)
370    Y(0)=.5*Delxi*Muy
380    Ns=0
390    Ns=Ns+1                   !  LOOP ON xi
400    Xi=Delxi*Ns               !  ARGUMENT xi OF CHAR. FN.
410    Txi=Xi*2.                 !  CALCULATION OF
420    Pr=1.                     !  CHARACTERISTIC
430    Pi=0.                     !  FUNCTION fy(xi)
440    FOR Ks=1 TO Kdc
450    Te=Txi*D(Ks)
460    Pe=Pr+Te*Pi
470    Pi=Pi-Te*Pr
480    Pr=Pe
490    NEXT Ks
500    CALL Sqr(Pr,Pi,Sr,Si)
510    De=Sr*Sr+Si*Si
520    Fyr=Sr/De
530    Fyi=-Si/De
540    Ro=R
550    R=ATN(Fyi/Fyr)
560    IF ABS(R-Ro)>1.6 THEN P=-P
570    IF P>0. THEN 600
```

```
580     Fyr=-Fyr
590     Fyi=-Fyi
600     Ms=Ns MODULO M          !  COLLAPSING
610     A=Fyr/Ns
620     B=Fyi/Ns
630     X(Ms)=X(Ms)+A
640     Y(Ms)=Y(Ms)+B
650     IF A*A+B*B<Tol THEN 670
660     GOTO 390
670     CALL Fft14(M,Cos(*),X(*),Y(*))
680     GCLEAR
690     WINDOW 0,M,-18,0
700     LINE TYPE 3
710     GRID M/8,1
720     LINE TYPE 1
730     FOR Ks=0 TO M-1
740     T=Y(Ks)/PI-Ks/M
750     Y(Ks)=Pr=.5+T                     ! EXCEEDANCE PROBABILITY IN Y(*)
760     IF Pr>=1.E-16 THEN Y=LGT(Pr)!Y(Ks)=PROB(x>2 PI Ks/(M Delxi)-Bs)
770     IF Pr<=-1.E-16 THEN Y=-32.-LGT(-Pr)
780     IF ABS(Pr)<1.E-16 THEN Y=-16.
790     PLOT Ks,Y
800     NEXT Ks
810     PENUP
820     IF I=Nr THEN 940
830     PRINT I;
840     INPUT "SCALE FFT SIZE BY 1,2,4,8,...:",Im
850     Mm=MIN(M*Im,16384)
860     IF Mm=M THEN 940
870     Delxi=Delxi*M/Mm
880     M=Mm
890     REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
900     A=2.*PI/M
910     FOR Ms=0 TO M/4
920     Cos(Ms)=COS(A*Ms)       !  QUARTER-COSINE TABLE; <= 4096
930     NEXT Ms
940     FOR Ks=0 TO 127
950     Pr(I,Ks)=Y(Ks)          !  STORE FOR ROC PLOT
960     NEXT Ks
970     NEXT I
980     FOR Ks=0 TO 127
990     IF Pr(0,Ks)<.1 THEN 1010
1000    NEXT Ks
1010    Ms=Ks-1
1020    FOR Ks=Ms TO 127
1030    IF Pr(0,Ks)<1E-10 THEN 1050
1040    NEXT Ks
1050    Ns=Ks
1060      ASSIGN #1 TO "ROC"
1070    PRINT #1;Ms,Ns,Nr
1080    PRINT #1;Pr(*)
1090    ASSIGN #1 TO *
1100    DIM Pf(1:14),Pd(1:18)
1110    DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4
1120    DATA .001,.002,.005,.01,.02,.05,.1
1130    READ Pf(*)
```

```
1140    DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7
1150    DATA .8,.9,.95,.98,.99,.995,.998,.999
1160    READ Pd(*)
1170    FOR I=1 TO 14
1180    Pf(I)=FNInvphi(Pf(I))
1190    NEXT I
1200    FOR I=1 TO 18
1210    Pd(I)=FNInvphi(Pd(I))
1220    NEXT I
1230    GCLEAR
1240    X1=Pf(1)
1250    X2=Pf(14)
1260    Y1=Pd(1)
1270    Y2=Pd(18)
1280    WINDOW X1,X2,Y1,Y2
1290    LINE TYPE 3
1300    FOR I=1 TO 14
1310    MOVE Pf(I),Y1
1320    DRAW Pf(I),Y2
1330    NEXT I
1340    FOR I=1 TO 18
1350    MOVE X1,Pd(I)
1360    DRAW X2,Pd(I)
1370    NEXT I
1380    PENUP
1390    LINE TYPE 1
1400    FOR Ks=Ms TO Ns
1410    Pr(0,Ks)=FNInvphi(Pr(0,Ks)) ! FALSE ALARM PROBABILITY
1420    NEXT Ks
1430    FOR I=1 TO Nr
1440    FOR Ks=Ms TO Ns
1450    X=Pr(0,Ks)
1460    Pr=Pr(I,Ks)
1470    IF Pr>.9999 THEN 1510
1480    Pr(I,Ks)=Y=FNInvphi(Pr)
1490    PLOT X,Y
1500    IF Pr<.01 THEN 1520
1510    NEXT Ks
1520    PENUP
1530    NEXT I
1540    PAUSE
1550    END
1560    !
1570    DEF FNArg(X,Y)          !  PRINCIPAL ARG(Z)
1580    IF X=0. THEN RETURN .5*PI*SGN(Y)
1590    A=ATN(Y/X)
1600    IF X>0. THEN RETURN A
1610    IF Y<0. THEN RETURN A-PI
1620    RETURN A+PI
1630    FNEND
1640    !
1650    SUB Sqr(X,Y,U,V)        !  PRINCIPAL SQR(Z)
1660    F=SQR(SQR(X*X+Y*Y))
1670    T=.5*FNArg(X,Y)
1680    U=F*COS(T)
1690    V=F*SIN(T)
1700    SUBEND
1710    !
```

```
1720    DEF FNInvphi(X)         !  AMS 55, 26.2.23
1730    IF X=.5 THEN RETURN 0.
1740    P=MIN(X,1.-X)
1750    P=MAX(P,1.E-20)
1760    T=-LOG(P)
1770    T=SQR(T+T)
1780    P=1.+T*(1.432788+T*(.189269+T*.001308))
1790    P=T-(2.515517+T*(.802853+T*.010328))/P
1800    IF X<.5 THEN P=-P
1810    RETURN P
1820    FNEND
1830    !
1840    SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1850    DOUBLE Log2n,N1,N2,N3,N4,J,K    !  INTEGERS < 2^31 = 2,147,483,648
1860    DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1870    IF N=1 THEN SUBEXIT
1880    IF N>2 THEN 1960
1890    A=X(0)+X(1)
1900    X(1)=X(0)-X(1)
1910    X(0)=A
1920    A=Y(0)+Y(1)
1930    Y(1)=Y(0)-Y(1)
1940    Y(0)=A
1950    SUBEXIT
1960    A=LOG(N)/LOG(2.)
1970    Log2n=A
1980    IF ABS(A-Log2n)<1.E-8 THEN 2010
1990    PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
2000    PAUSE
2010    N1=N/4
2020    N2=N1+1
2030    N3=N2+1
2040    N4=N3+N1
2050    FOR I1=1 TO Log2n
2060    I2=2^(Log2n-I1)
2070    I3=2*I2
2080    I4=N/I3
2090    FOR I5=1 TO I2
2100    I6=(I5-1)*I4+1
2110    IF I6<=N2 THEN 2150
2120    A1=-Cos(N4-I6-1)
2130    A2=-Cos(I6-N1-1)
2140    GOTO 2170
2150    A1=Cos(I6-1)
2160    A2=-Cos(N3-I6-1)
2170    FOR I7=0 TO N-I3 STEP I3
2180    I8=I7+I5-1
2190    I9=I8+I2
2200    T1=X(I8)
2210    T2=X(I9)
2220    T3=Y(I8)
2230    T4=Y(I9)
```

```
2240    A3=T1-T2
2250    A4=T3-T4
2260    X(I8)=T1+T2
2270    Y(I8)=T3+T4
2280    X(I9)=A1*A3-A2*A4
2290    Y(I9)=A1*A4+A2*A3
2300    NEXT I7
2310    NEXT I5
2320    NEXT I1
2330    I1=Log2n+1
2340    FOR I2=1 TO 14
2350    L(I2-1)=1
2360    IF I2>Log2n THEN 2380
2370    L(I2-1)=2^(I1-I2)
2380    NEXT I2
2390    K=0
2400    FOR I1=1 TO L(13)
2410    FOR I2=I1 TO L(12) STEP L(13)
2420    FOR I3=I2 TO L(11) STEP L(12)
2430    FOR I4=I3 TO L(10) STEP L(11)
2440    FOR I5=I4 TO L(9) STEP L(10)
2450    FOR I6=I5 TO L(8) STEP L(9)
2460    FOR I7=I6 TO L(7) STEP L(8)
2470    FOR I8=I7 TO L(6) STEP L(7)
2480    FOR I9=I8 TO L(5) STEP L(6)
2490    FOR I10=I9 TO L(4) STEP L(5)
2500    FOR I11=I10 TO L(3) STEP L(4)
2510    FOR I12=I11 TO L(2) STEP L(3)
2520    FOR I13=I12 TO L(1) STEP L(2)
2530    FOR I14=I13 TO L(0) STEP L(1)
2540    J=I14-1
2550    IF K>J THEN 2620
2560    A=X(K)
2570    X(K)=X(J)
2580    X(J)=A
2590    A=Y(K)
2600    Y(K)=Y(J)
2610    Y(J)=A
2620    K=K+1
2630    NEXT I14
2640    NEXT I13
2650    NEXT I12
2660    NEXT I11
2670    NEXT I10
2680    NEXT I9
2690    NEXT I8
2700    NEXT I7
2710    NEXT I6
2720    NEXT I5
2730    NEXT I4
2740    NEXT I3
2750    NEXT I2
2760    NEXT I1
2770    SUBEND
```

```
10   ! SIMULATION FOR ONE-POLE PROCESSES
20     As=.85                    !  INPUT SIGNAL PARAMETER
30     Bs=.77                    !  INPUT NOISE  PARAMETER; LINE 160
40     Cs=1.13                   !  FILTER PARAMETER
50     R=.7                      !  INPUT POWER SNR
60     Ka=4
70     Kb=11
80     Kc=2
90     Kd=16
100    Bins=1000
110    Num=1E6                   !  NUMBER OF TRIALS
120    Deltaz=.25                !  BIN SIZE FOR OUTPUT Z
130    DOUBLE Ka,Kb,Kc,Kd,Bins,Num,It,K,M
140    DIM E(100),S(100),Yn(100),Ys(100),T(1000)
150    Ea=EXP(-As)
160    Eb=EXP(-Bs)               !  WHITE NOISE: FOR Bs=inf, SET Eb=0
170    Ec=EXP(-Cs)
180    Fa=SQR(R*(1.-Ea*Ea))
190    Fb=SQR(1.-Eb*Eb)
200    REDIM E(0:Kd-Ka)
210    E(0)=Cs
220    FOR K=1 TO Kd-Ka
230    E(K)=E(K-1)*Ec
240    NEXT K
250    REDIM S(-30:Kd),Yn(Kc-1:Kd),Ys(Ka:Kd),T(1:Bins)
260    FOR It=1 TO Num
270    S(-30)=Nk=Yk=0.
280    FOR K=-29 TO Kc-1         !  ALLOW STEADY STATE
290    R1=2.*RND-1.              !  BOX-MULLER
300    R2=2.*RND-1.
310    R3=R1*R1+R2*R2
320    IF R3>1. THEN 290
330    R3=SQR(-2.*LOG(R3)/R3)
340    R1=R1*R3
350    R2=R2*R3
360    S(K)=Fa*R1+Ea*S(K-1)  !  FILTER INPUT  SIGNAL
370    Nk=Fb*R2+Eb*Nk            !  FILTER INPUT  NOISE
380    Yk=Cs*Nk+Ec*Yk            !  FILTER OUTPUT NOISE
390    NEXT K
400    Yn(Kc-1)=Yk
410    FOR K=Kc TO Kd
420    R1=2.*RND-1.
430    R2=2.*RND-1.
440    R3=R1*R1+R2*R2
450    IF R3>1. THEN 420
460    R3=SQR(-2.*LOG(R3)/R3)
470    R1=R1*R3
480    R2=R2*R3
490    M=K-1
```

```
500     S(K)=Fa*R1+Ea*S(M)   !  FILTER INPUT  SIGNAL
510     Nk=Fb*R2+Eb*Nk       !  FILTER INPUT  NOISE
520     Yn(K)=Cs*Nk+Ec*Yn(M) !  FILTER OUTPUT NOISE
530     NEXT K
540     FOR K=Ka TO Kd
550     S=0.
560     FOR M=Ka TO MIN(K,Kb)
570     S=S+E(K-M)*S(M)      !  FILTER OUTPUT SIGNAL
580     NEXT M
590     Ys(K)=S
600     NEXT K
610     Z=0.
620     FOR K=Kc TO Ka-1
630     T=Yn(K)
640     Z=Z+T*T
650     NEXT K
660     FOR K=Ka TO Kd
670     T=Ys(K)+Yn(K)
680     Z=Z+T*T
690     NEXT K
700     K=INT(Z/Deltaz)+1
710     K=MIN(K,Bins)
720     T(K)=T(K)+1.
730     NEXT It
740     FOR K=Bins-1 TO 1 STEP -1
750     T(K)=T(K)+T(K+1)
760     NEXT K
770     MAT T=T/(Num)
780     GINIT
790     PLOTTER IS "GRAPHICS"
800     GRAPHICS ON
810     WINDOW 0,Bins,-5,0
820     GRID Bins/4,1
830     FOR K=1 TO Bins
840     T=T(K)
850     IF T>0. THEN 870
860     GOTO 890
870     PLOT K,LGT(T)
880     NEXT K
890     PENUP
900     FOR K=Bins TO 1 STEP -1
910     T=1.-T(K)
920     IF T>0. THEN 940
930     GOTO 960
940     PLOT K,LGT(T)
950     NEXT K
960     PENUP
970     PAUSE
980     DUMP GRAPHICS
990     END
```

APPENDIX E. PROGRAM FOR COLORED INPUT NOISE

One program is listed in this appendix. It computes the covariance matrices and eigenvalues as described in appendix D, but now for a colored noise input. The program for computing the receiver operating characteristics from the eigenvalues is identical to that listed above and therefore has not been repeated here.

```
 10   ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES
 20   ! STORE EIGENVALUES IN "EIG" IN LINE 2260
 30     Ka=-10                 !  INPUT SIGNAL START
 40     Kb=5                   !  INPUT SIGNAL END; Kb >= Ka
 50     Kc=0                   !  ACCUMULATOR  START
 60     Kd=5                   !  ACCUMULATOR  END; Kd >= Kc,Ka
 70     Delta=.2               !  TIME SAMPLING INCREMENT (SECONDS)
 80     J=3                    !  NUMBER OF SIGNAL COMPONENTS
 90     DATA 11.,-48.,75.      !  SIGNAL SCALINGS (WE SET SUM = 1)
100     DATA 1.,.125,.066667   !  SIGNAL TIME CONSTANTS (SECONDS)
110     M=2                    !  NUMBER OF NOISE COMPONENTS
120     DATA 39.,60.           !  NOISE SCALINGS (WE SET SUM = 1)
130     DATA .2,.4             !  NOISE TIME CONSTANTS (SECONDS)
140     N=4                    !  NUMBER OF FILTER COMPONENTS
150     DATA 1.,2.,3.,4.       !  FILTER SCALINGS (ARBITRARY)
160     DATA .3,.5,.7,.9       !  FILTER TIME CONSTANTS (SECONDS)
170     Nr=20                  !  NUMBER OF SIGNAL-TO-NOISE RATIOS
180     DATA 0,1,2,3,4,5,6,7,8,9,10         !  INPUT SNRS IN DB
190     DATA 11,12,13,14,15,16,17,18,19
200     IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 230
210     PRINT "PROBLEM WITH PARAMETERS"
220     STOP
230     DIM Alpha(10),A(10),Beta(10),B(10),Psi(10),C(10),R(20)
240     DIM W(100),Pc(10),Ec(10),Pcv(10),Eb(10),Chb(10),Shb(10)
250     DIM Chc(10),Shc(10),Mu(10),Gamma(10),Cn(200),Ea(10)
260     DIM E(0,1000),Gca(100),Gcda(100),Gcb(100),Gbda(100)
270     DIM Cs(5000),Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
280     DOUBLE Ka,Kb,Kc,Kd,J,M,N,Nr,Ns,Ks,Ms,Kdc,Ko
290     DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
300     REDIM Alpha(1:J),A(1:J)
310     READ Alpha(*),A(*)
320     MAT Alpha=Alpha/(SUM(Alpha))
330     MAT A=(Delta)/A
340     REDIM Beta(1:M),B(1:M)
350     READ Beta(*),B(*)
360     MAT Beta=Beta/(SUM(Beta))
370     MAT B=(Delta)/B
380     REDIM Psi(1:N),C(1:N)
390     READ Psi(*),C(*)
400     MAT C=(Delta)/C
410     REDIM R(1:Nr)
420     READ R(*)
430     REDIM W(Kc:Kd)
440     CALL Weights(Kc,Kd,W(*))
450     MAT W=SQR(W)
```

```
1070    Cn(Ks)=Cn(Ks)+Mu*P
1080    NEXT Ks
1090    NEXT Ms
1100    FOR Ns=1 TO N
1110    Ga=Gamma(Ns)
1120    E=Ec(Ns)
1130    P=1.
1140    Cn(0)=Cn(0)+Ga
1150    FOR Ks=1 TO Kdc
1160    P=P*E
1170    Cn(Ks)=Cn(Ks)+Ga*P
1180    NEXT Ks
1190    NEXT Ns
1200    FOR Js=1 TO J          !  FILTER OUTPUT SIGNAL COVARIANCE
1210    A(Js)=EXP(A(Js))
1220    NEXT Js
1230    Ko=MAX(Ka,Kc)
1240    REDIM Ea(1:N),E(1:N,Ko:Kd)
1250    FOR Ns=1 TO N
1260    E=Ec(Ns)
1270    Ea(Ns)=1./E
1280    E(Ns,Ko)=EXP(-C(Ns)*Ko)
1290    FOR Ks=Ko+1 TO Kd
1300    E(Ns,Ks)=E(Ns,Ks-1)*E
1310    NEXT Ks
1320    NEXT Ns
1333    L1=MIN(Ko,Kb)+1
1340    L2=MIN(Kd,Kb)+1
1350    REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
1360    L=Kd-Ko+1
1370    REDIM Cs(1:L*(L+1)/2)
1380    L11=L1+1
1390    FOR Js=1 TO J
1400    Al=Alpha(Js)
1410    A=A(Js)
1420    A1=A*A-1.
1430    FOR Ns=1 TO N
1440    C=Ea(Ns)
1450    Ca=C*A
1460    Cda=C/A
1470    Ca1=1.-Ca
1480    Ac=A-C
1490    Ae=Al*Pc(Ns)
1500    Gca=Ca^Ka/Ca1
1510    Gca(L1)=Ca^L1/Ca1
1520    Gcda(L1)=Cda^L1/(1.-Cda)
1530    FOR Ks=L11 TO L2
1540    Gca(Ks)=Ca*Gca(Ks-1)
1550    Gcda(Ks)=Cda*Gcda(Ks-1)
1560    NEXT Ks
1570    FOR Qs=1 TO N
1580    B=Ea(Qs)
1590    Aee=Ae*Pc(Qs)
1600    Ba=B*A
1610    Bda=B/A
1620    Cb=C*B
1630    Ba1=1.-Ba
1640    Cb1=1.-Cb
1650    Gba=Ba^Ka/Ba1
1660    Ab=A-B
1670    F=B*A1/(Ab*Ba1)
```

```
1680    S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1690    Gcb(L1)=Cb^L1/Cb1
1700    Gbda(L1)=Bda^L1/(1.-Bda)
1710    FOR Ks=L11 TO L2
1720    Gcb(Ks)=Cb*Gcb(Ks-1)
1730    Gbda(Ks)=Bda*Gbda(Ks-1)
1740    NEXT Ks
1750    Ki=0
1760    FOR Ks1=Ko TO Kd
1770    K1=MIN(Ks1,Kb)+1
1780    S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1790    S3=Gca(K1)-Gca
1800    E=Aee*E(Ns,Ks1)
1810    FOR Ks2=Ks1 TO Kd
1820    Ki=Ki+1
1830    K2=MIN(Ks2,Kb)+1
1840    S=S2+Gbda(K2)*S3       !  SUM S
1850    Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*S
1860    NEXT Ks2
1870    NEXT Ks1
1880    NEXT Qs
1890    NEXT Ns
1900    NEXT Js
1910    L=Kd-Kc+1              !  TOTAL WEIGHTED COVARIANCE MATRIX
1920    REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1930    L11=Kc-1
1940    R=0.
1950    FOR I=0 TO Nr
1960    Ki=0
1970    IF I=0 THEN 1990
1980    R=10.^(R(I)*.1)        !  INPUT POWER SIGNAL-TO-NOISE RATIO
1990    FOR Ks1=Kc TO Kd
2000    W=W(Ks1)               !  WEIGHTS (w(k))
2010    FOR Ks2=Ks1 TO Kd
2020    IF Ks1>=Ko THEN 2050
2030    Cs=0.
2040    GOTO 2070
2050    Ki=Ki+1
2060    Cs=Cs(Ki)
2070    Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
2080    L1=Ks1-L11
2090    L2=Ks2-L11
2100    Cw(L1,L2)=Cw(L2,L1)=Pr
2110    NEXT Ks2
2120    NEXT Ks1
2130    CALL Svd(L,L,Cw(*),Ca(*),D(*))  !  EIGENVALUES
2140    MAT SORT D(*) DES
2150    IF D(L)>0. THEN 2180
2160    PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
2170    PAUSE
2180    PRINT "I =";I;"   CONDITION NUMBER =";D(1)/D(L)
2190    PRINT D(*)
2200    PRINT
2210    FOR Ks=1 TO L
2220    Eig(I,Ks)=D(Ks)        !  STORE EIGENVALUES
2230    NEXT Ks
2240    NEXT I
2250    MASS STORAGE IS ":CS80,7"
2260    ASSIGN #1 TO "EIG"
2270    PRINT #1;Nr,Kc,Kd,Eig(*)
2280    ASSIGN #1 TO *
2290    END
```

REFERENCES

[1] F. B. Hildebrand, **Methods of Applied Mathematics**, Prentice-Hall, Inc., New York, NY, 1954.

[2] A. H. Nuttall, **Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions**, NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

[3] A. H. Nuttall, **Operating Characteristics for Weighted Energy Detector with Gaussian Signals**, NUSC Technical Report 8753, Naval Underwater Systems Center, New London, CT, 16 July 1990.

NUSC Technical Document 8921
12 December 1991

Comparison of Two Kernels for the Modified Wigner Distribution Function

Chintana Griffin and Albert H. Nuttall

ABSTRACT

This document contains the lecture presentation of the paper entitled "Comparison of Two Kernels for the Modified Wigner Distribution Function," given at the Society of Photo-Optical Instrumentation Engineers International Symposium on Optical Applied Science and Engineering, Conference 1566 on Advanced Signal Processing Algorithms, Architectures, and Implementations, 21 - 26 July 1991, San Diego, California.

We compare the modified Wigner distribution functions obtained via the Choi-Williams kernel and its rotation, as well as by the tilted Gaussian kernel. Based on several commonly used examples, we demonstrate that the modified Wigner distribution obtained via the Gaussian kernel can minimize the artifacts more effectively and had the capability of selectively filtering out undesired components.

COMPARISON OF TWO KERNELS FOR THE MODIFIED WIGNER DISTRIBUTION FUNCTION

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OBJECTIVE

- COMPARE TILTED GAUSSIAN AND CHOI-WILLIAMS KERNELS IN TERMS OF MODIFIED WIGNER DISTRIBUTIONS FOR MULTI-COMPONENT SIGNALS.

CRITERIA FOR THE COMPARISON ARE

- CROSS-TERM SUPPRESSION
- DESIRED COMPONENT EXTRACTION

OUTLINE

- WIGNER DISTRIBUTION & COMPLEX AMBIGUITY FUNCTION
- MODIFIED WIGNER DISTRIBUTION
  - TILTED GAUSSIAN KERNEL
  - CHOI-WILLIAMS KERNEL
- ROTATION OF KERNEL
- EXAMPLES
- CONCLUSION

WIGNER DISTRIBUTION & COMPLEX AMBIGUITY FUNCTION

TCF: $$R(t,\tau) = s\left(t+\frac{\tau}{2}\right)s^*\left(t-\frac{\tau}{2}\right)$$

SCF: $$\Phi(\nu,f) = S\left(f+\frac{\nu}{2}\right)S^*\left(f-\frac{\nu}{2}\right)$$

WDF: $$W(t,f) = \int R(t,\tau)\exp(-j2\pi f\tau)\,d\tau = \int \Phi(\nu,f)\exp(j2\pi\nu t)\,d\nu$$

CAF: $$\chi(\nu,\tau) = \int R(t,\tau)\exp(-j2\pi\nu t)\,dt = \int \Phi(\nu,f)\exp(j2\pi f\tau)\,df$$

$$W(t,f) = \iint \chi(\nu,\tau)\exp(j2\pi t\nu)\exp(-j2\pi f\tau)\,d\tau\,d\nu$$

WIGNER DISTRIBUTION & COMPLEX AMBIGUITY FUNCTION

WDF		CAF
AUTO-TERMS	$\longleftrightarrow$	CONCENTRATED NEAR ORIGIN
CROSS-TERMS	$\longleftrightarrow$	SHIFTED AWAY FROM ORIGIN

THE AUTO-TERMS AND CROSS-TERMS IN WDF, (t,f) DOMAIN, ARE ESSENTIALLY SEPARATED IN CAF, (ν,τ) DOMAIN. THE SEPARATION OF THE AUTO-TERMS AND CROSS-TERMS IN CAF DOMAIN MAKES (ν,τ) DOMAIN IDEAL FOR KERNEL DESIGN AND FILTERING.

MODIFIED TIME-FREQUENCY REPRESENTATIONS

modified CAF:

$$\hat{\chi}(\nu,\tau) = \chi(\nu,\tau)\, \tilde{v}(\nu,\tau)$$

modified SCF:

$$\hat{\Phi}(\nu,f) = \Phi(\nu,f) \overset{f}{\oplus} \tilde{V}(\nu,f)$$

modified TCF:

$$\hat{R}(t,\tau) = R(t,\tau) \overset{t}{\oplus} v(t,\tau)$$

modified WDF:

$$\hat{W}(t,f) = W(t,f) \overset{tf}{\oplus} V(t,f)$$

KERNEL IN EACH DOMAIN

SCF: $\tilde{V}(\nu,f) = \int \tilde{v}(\nu,\tau) \exp(-j2\pi f\tau)\, d\tau$

TCF: $v(t,\tau) = \int \tilde{v}(\nu,\tau) \exp(j2\pi \nu t)\, d\nu$

WDF: $V(t,f) = \int v(t,\tau) \exp(-j2\pi f\tau)\, d\tau$

$= \int \tilde{V}(\nu,f) \exp(j2\pi \nu t)\, d\nu$

$= \iint \tilde{v}(\nu,\tau) \exp(j2\pi\nu t - j2\pi f\tau)\, d\nu d\tau$

$\overset{X}{\oplus}$ **DENOTES CONVOLUTION ON X**

TILTED GAUSSIAN KERNEL

CAF: $$\tilde{v}(\nu,\tau) = \exp\left[-\pi\left(\frac{\nu^2}{B^2}+\frac{\tau^2}{D^2}+2r\frac{\nu}{B}\frac{\tau}{D}\right)\right]$$

WDF: $$V(t,f) = \frac{BD}{\sqrt{1-r^2}}\exp\left[-\frac{\pi}{1-r^2}(B^2t^2+D^2f^2+2rBtDf)\right]$$

ARBITRARY B,D , AND r

DIMENSIONLESS TILT PARAMETER $|r|<1$

CHOI-WILLIAMS KERNEL

CAF: $\tilde{v}(\nu,\tau) = \exp(-\nu^2\tau^2/\sigma^2), \qquad \sigma>0$

WDF:

$$V(t,f) = 2\pi^{1/2}\sigma\int_{0^+}^{\infty}\cos(2\pi\nu t)\exp(-\pi^2\sigma^2 f^2/\nu^2)\frac{d\nu}{\nu}$$

$$= 2\pi^{1/2}\sigma\int_{0^+}^{\infty}\cos(2\pi f\tau)\exp(-\pi^2\sigma^2 t^2/\tau^2)\frac{d\tau}{\tau},$$

$t\neq0,\ f\neq0$

ROTATION OF KERNEL

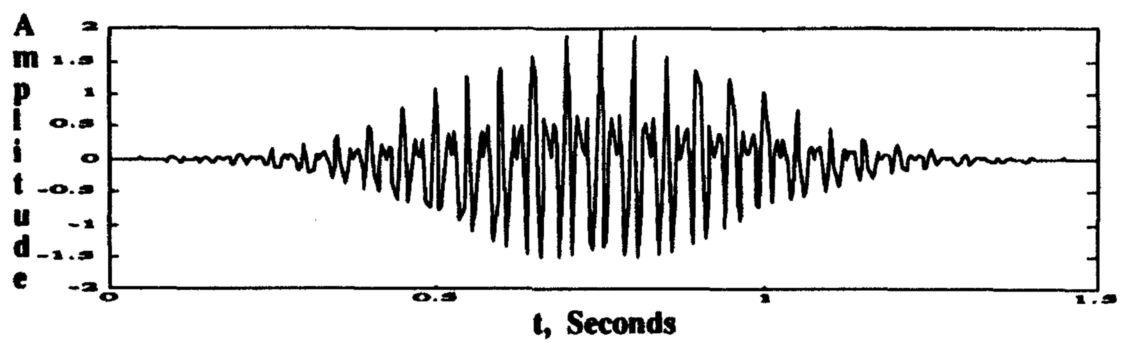
EXPRESS $\tilde{v}$ IN TERMS OF A NORMALIZED FUNCTION $\tilde{u}$:

$$\tilde{v}(\nu,\tau) = \tilde{u}(\nu/B, \tau/D)$$

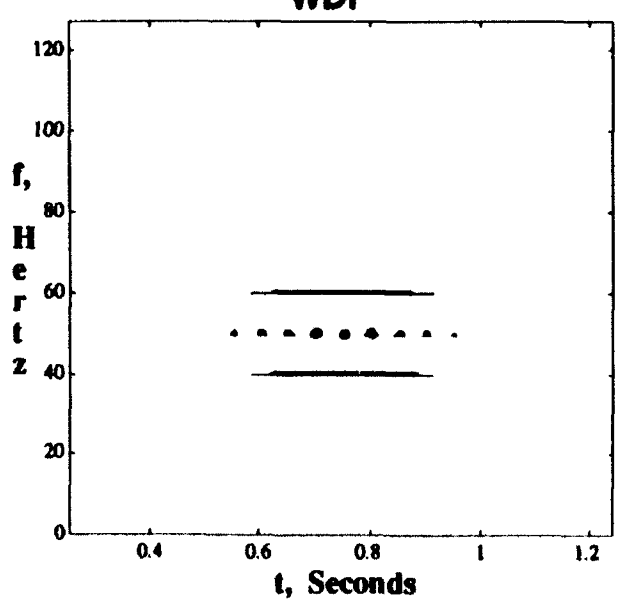
LET θ BE THE ANGLE OF ROTATION IN THE ν/B, τ/D PLANE, C=COS(θ), S=sin(θ). THEN THE ROTATED WEIGHTING IS

$$\begin{aligned}\tilde{r}(\nu,\tau) &= \tilde{u}\left(C\frac{\nu}{B} + S\frac{\tau}{D}, C\frac{\tau}{D} - S\frac{\nu}{B}\right)\\ &= \tilde{v}\left(C\nu + S\frac{B}{D}\tau, C\tau - S\frac{D}{B}\nu\right)\end{aligned}$$

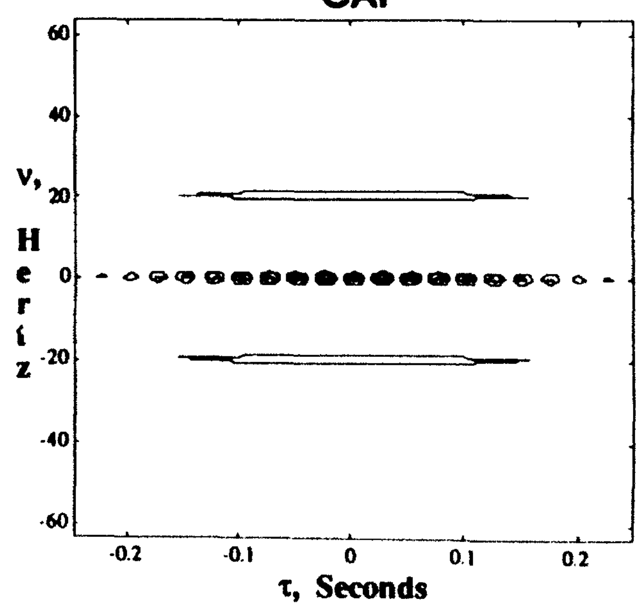
EXAMPLE 1: TWO AM SINEWAVES



WDF

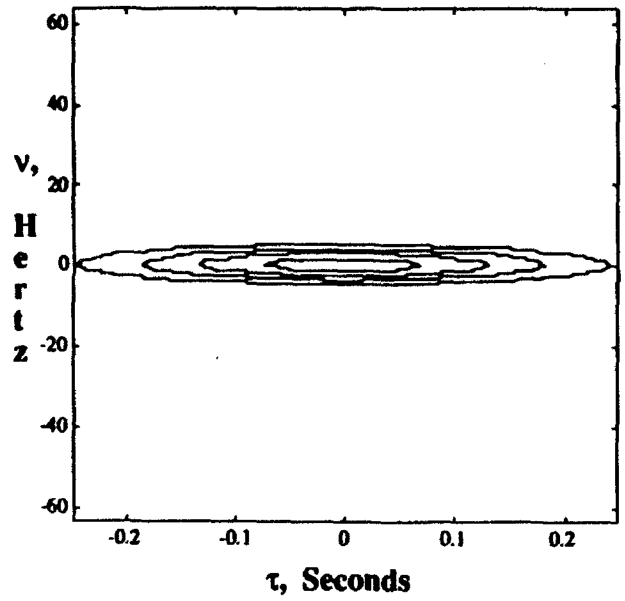


CAF

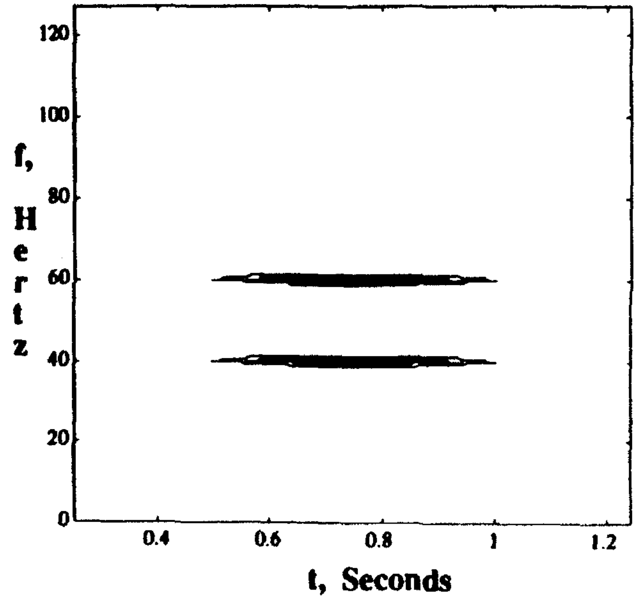


EXAMPLE 1: TWO AM SINEWAVES

GAUSSIAN KERNEL

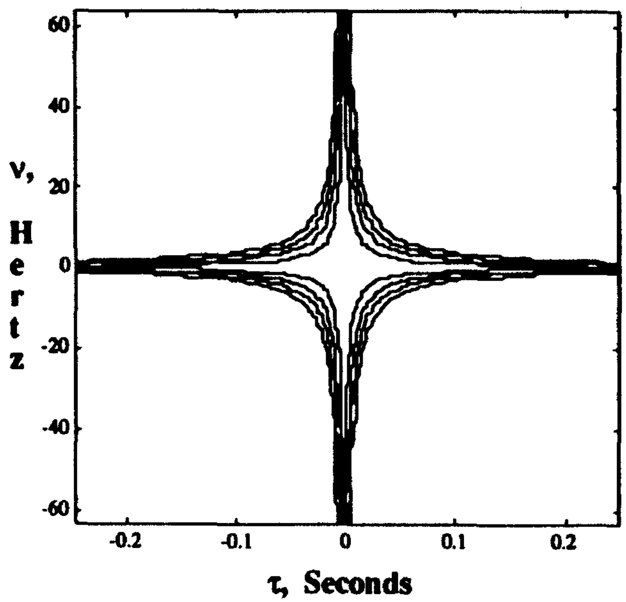


G-WDF

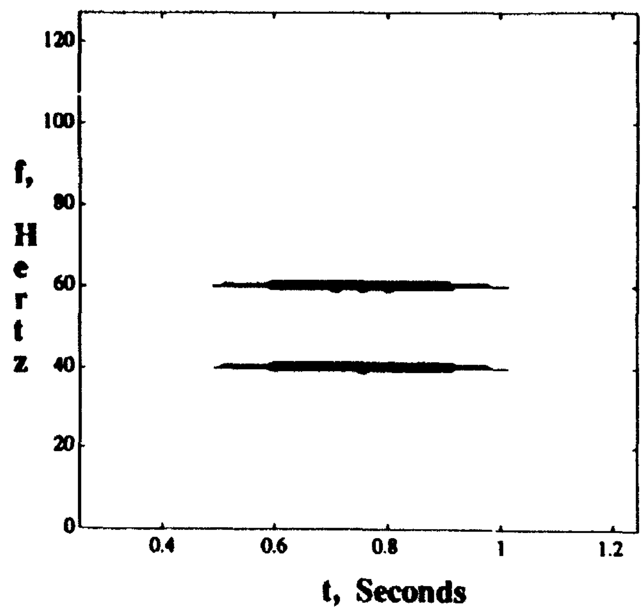


EXAMPLE 1: TWO AM SINEWAVES

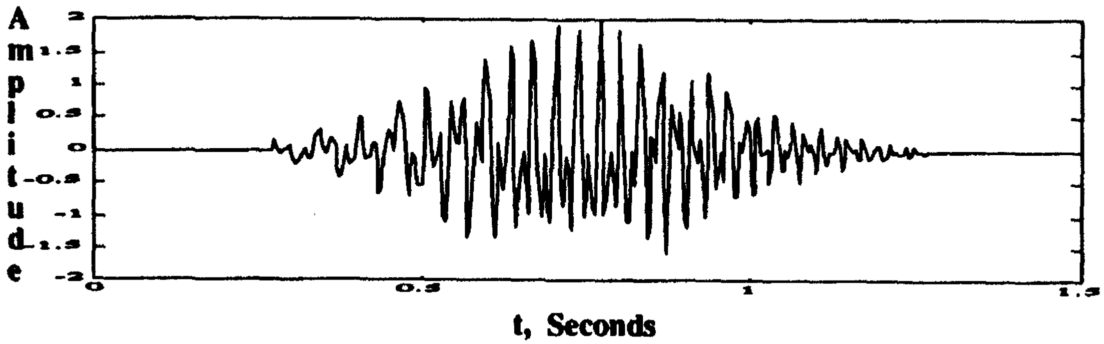
CHOI-WILLIAMS KERNEL

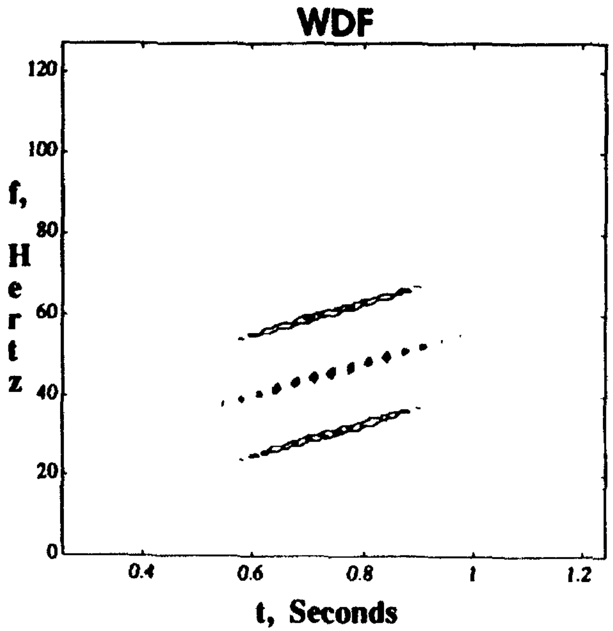


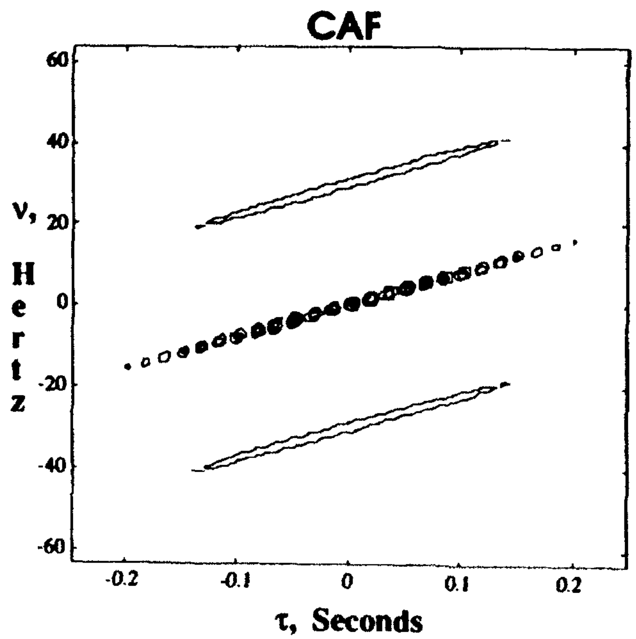
C-WDF



EXAMPLE 2: TWO PARALLEL LFMS

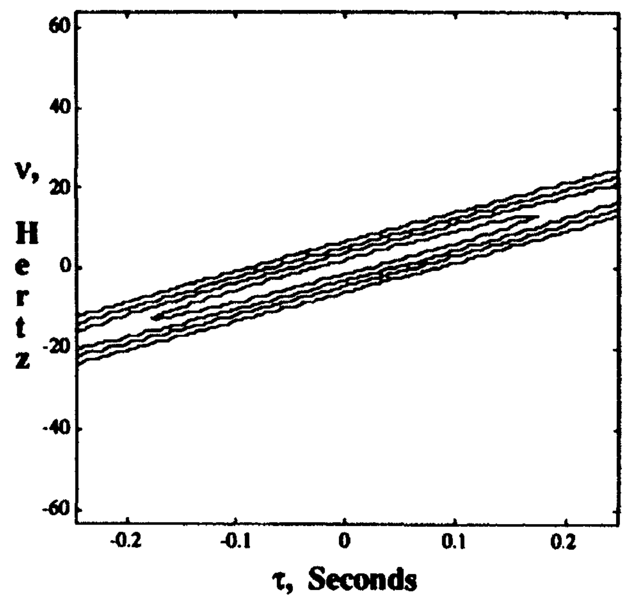




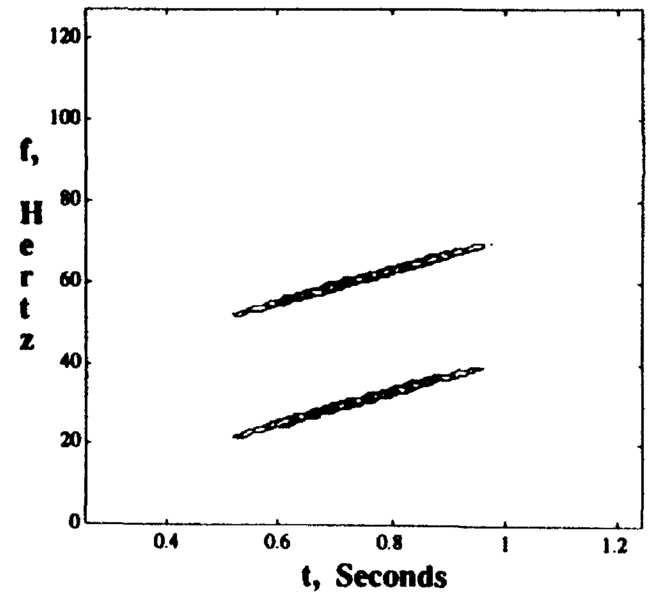


EXAMPLE 2: TWO PARALLEL LFMS

TILTED GAUSSIAN KERNEL

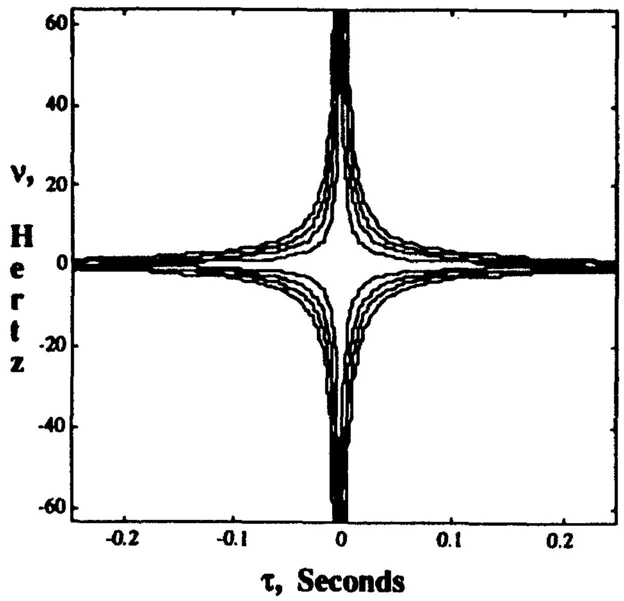


G-WDF



EXAMPLE 2: TWO PARALLEL LFMS

CHOI-WILLIAMS KERNEL



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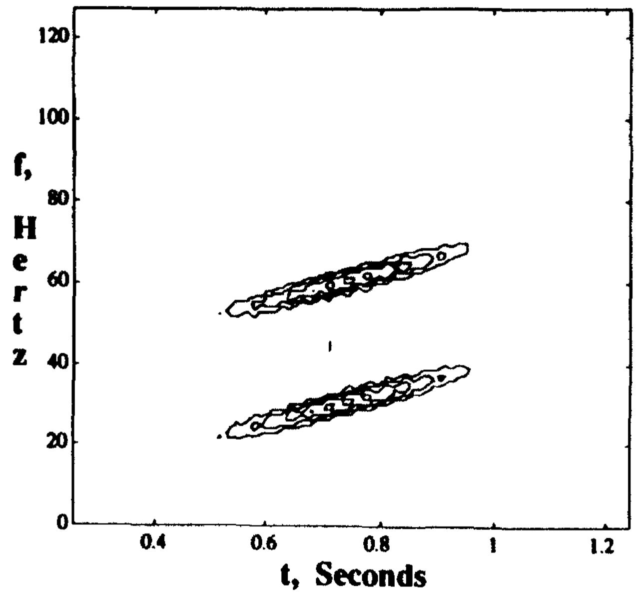
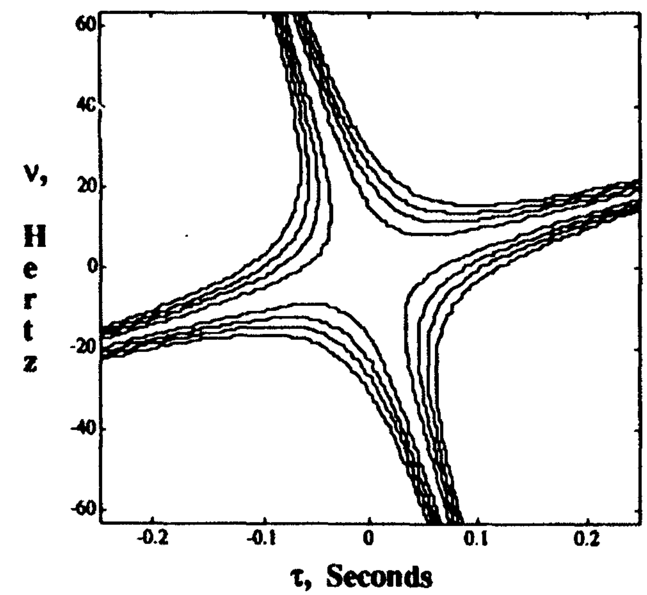


EXAMPLE 2: TWO PARALLEL LFMS

ROTATED CHOI-WILLIAMS KERNEL



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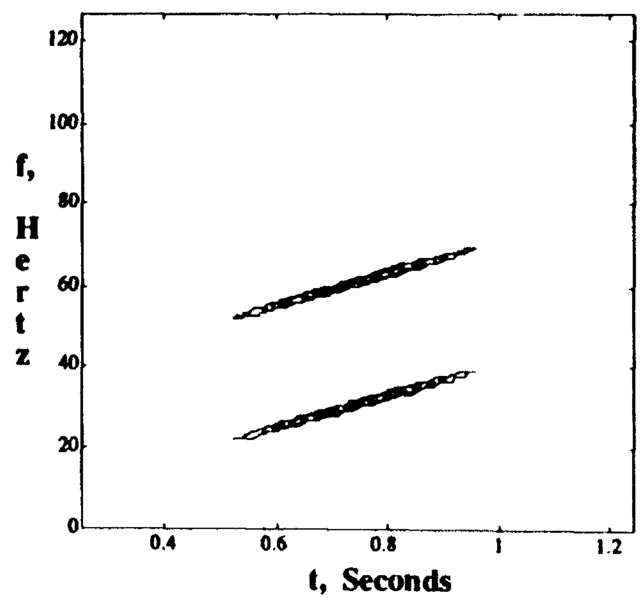
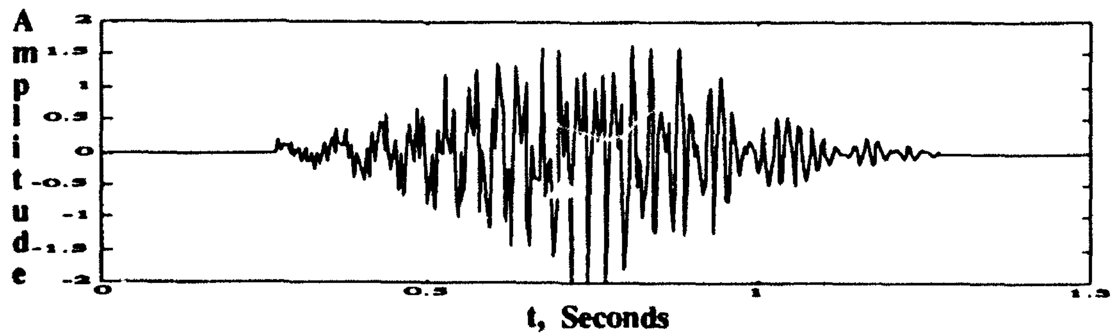
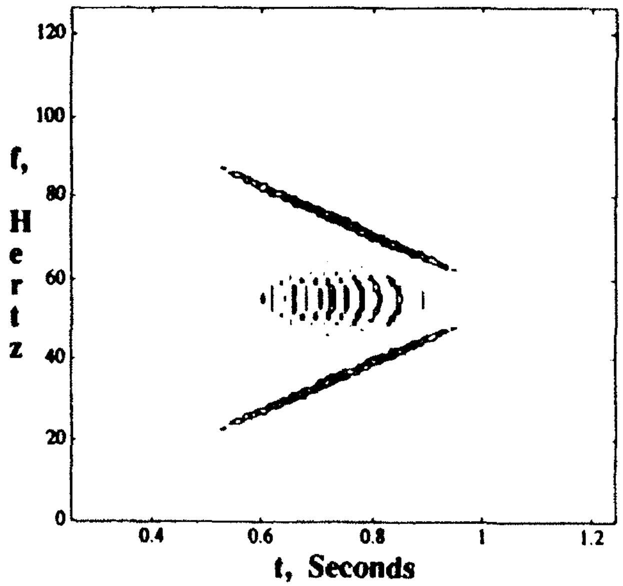


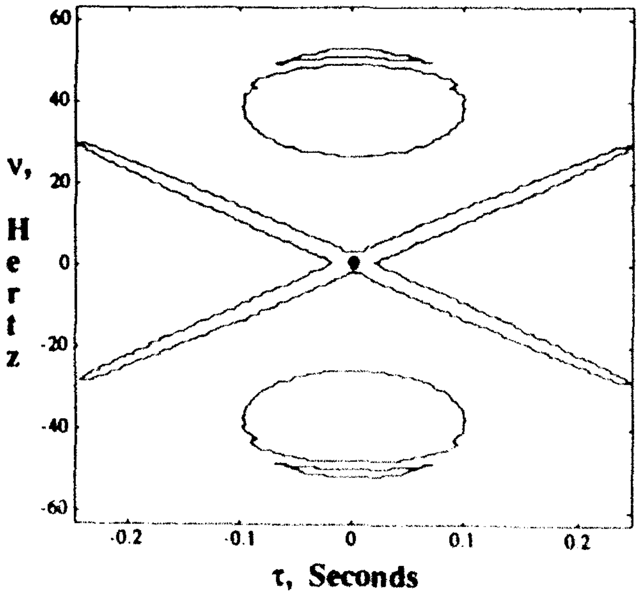
EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS



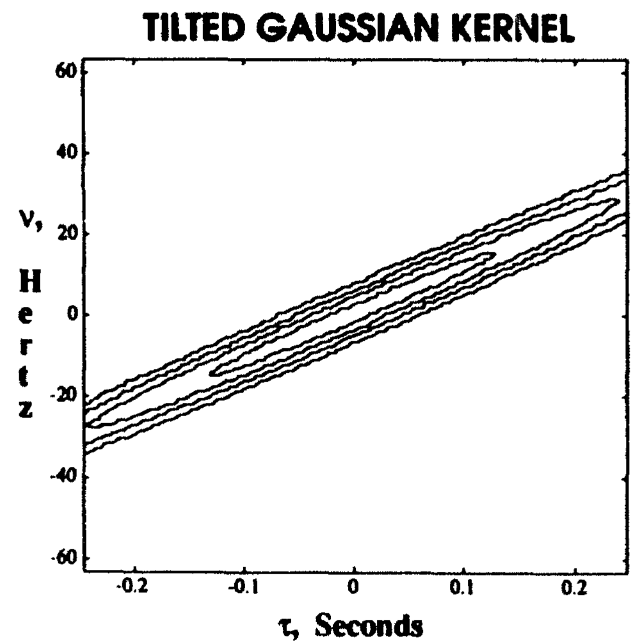
WDF

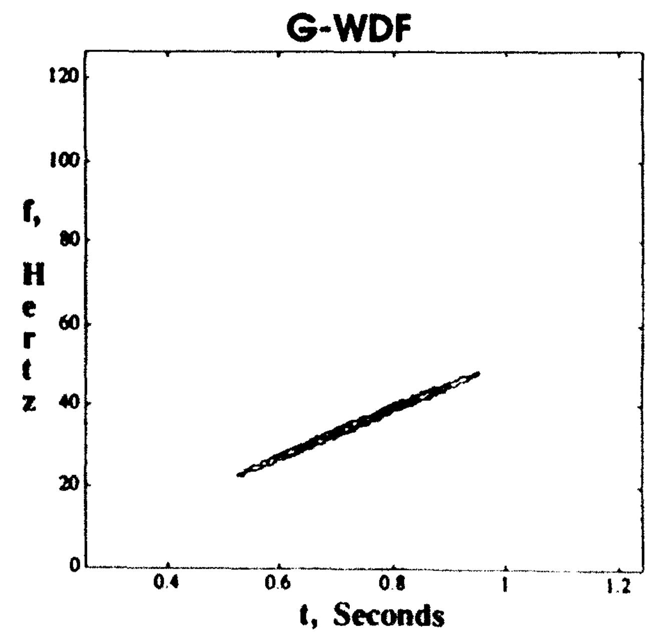


CAF



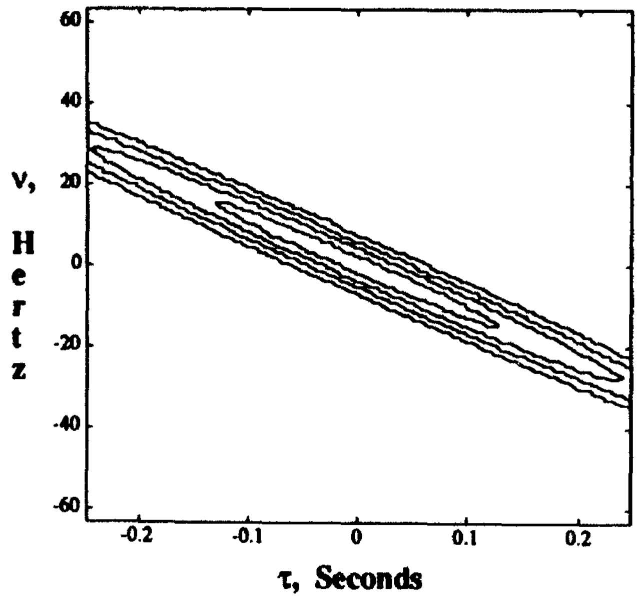
EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS





EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

TILTED GAUSSIAN KERNEL



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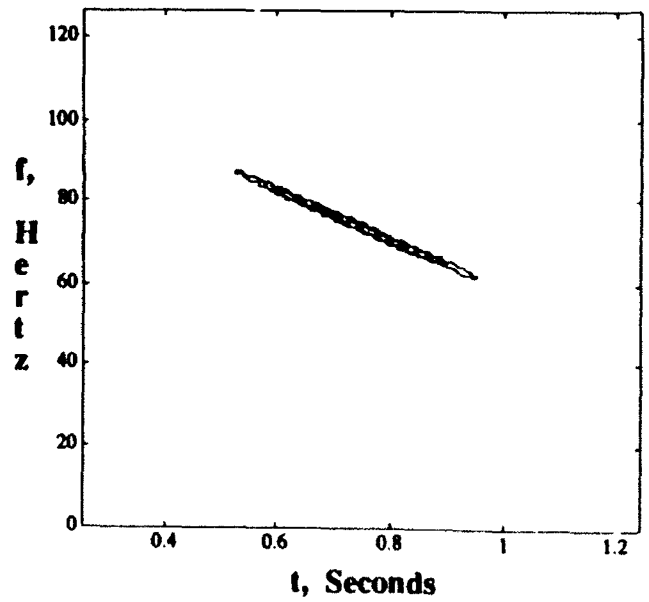
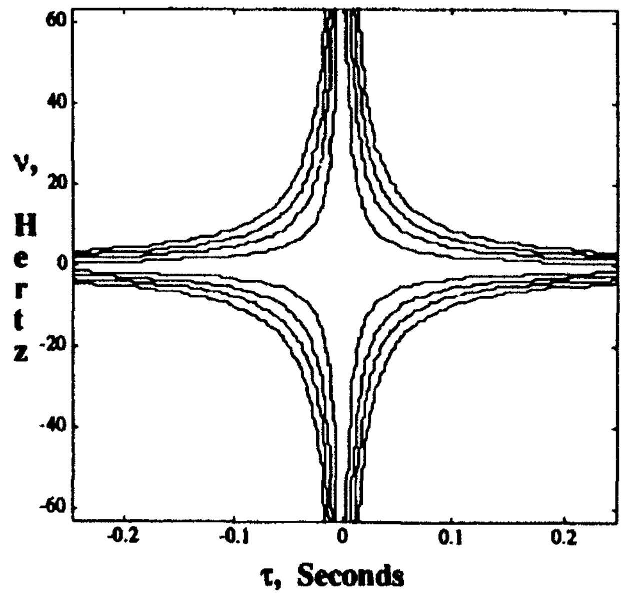


EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

CHOI-WILLIAMS KERNEL



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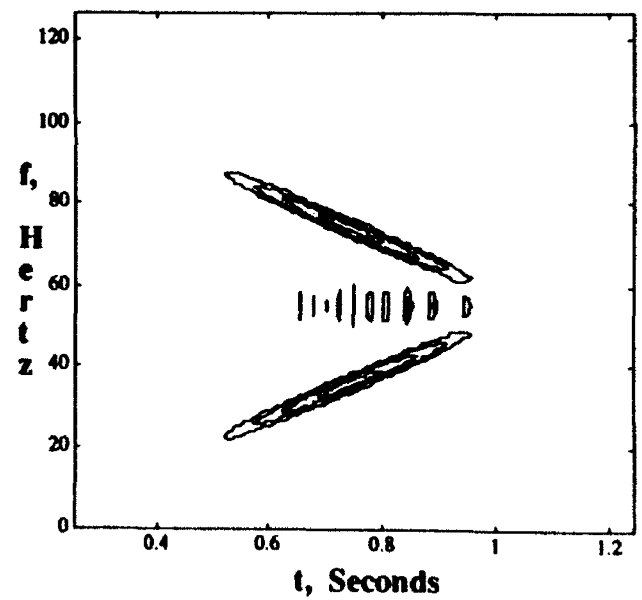
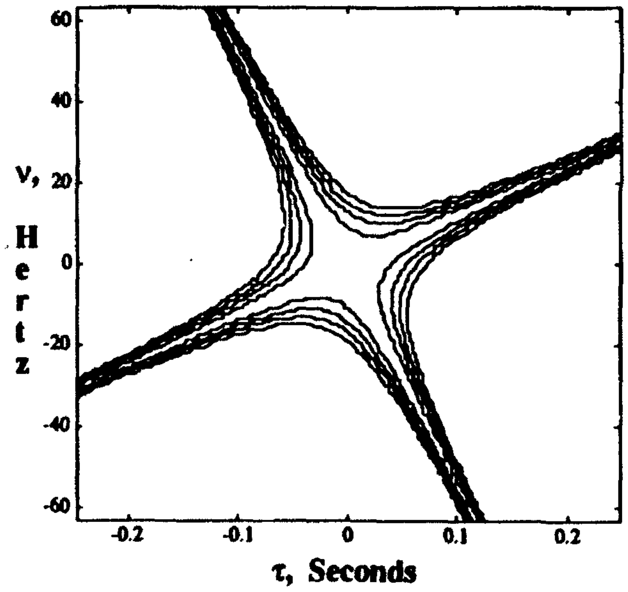
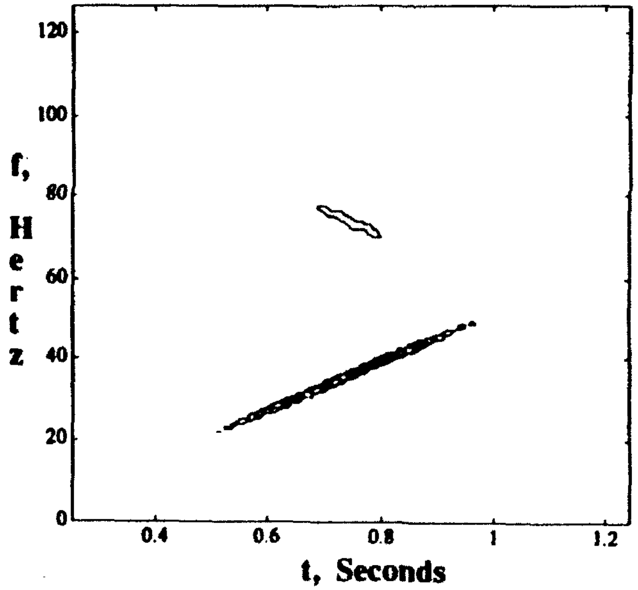


EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

ROTATED CHOI-WILLIAMS KERNEL



C-WDF



CONCLUSIONS

TILTED GAUSSIAN KERNEL

- EFFECTIVELY SUPPRESSES THE CROSS-TERMS AND RECOVERS THE AUTO-TERMS IN ALL CASES CONSIDERED HERE
- EFFECTIVELY EXTRACTS SELECTED COMPONENTS, PROVIDED THE ENERGY DISTRIBUTIONS DO NOT SIGNIFICANTLY OVERLAP IN THE (t,f) DOMAIN

CHOI-WILLIAMS KERNEL

- EFFECTIVELY SUPPRESSES THE CROSS-TERMS AND RECOVERS THE AUTO-TERMS ONLY FOR THE CASE WHERE THE SLOPES OF THE LINES OF INSTANTANEOUS FREQUENCY ARE ZERO; I.E. NO FM
- HAS NO CAPABILITY TO EXTRACT SELECTED COMPONENTS

CONCLUSIONS

ROTATED CHOI-WILLIAMS KERNEL

- SUBSTANTIALLY IMPROVES THE CROSS-TERM SUPPRESSION AND THE AUTO-TERM RECOVERY FOR THE CASE WHERE THE SLOPES OF THE LINES OF INSTANTANEOUS FREQUENCY ARE NON-ZERO
- EXTRACTION OF SELECTED COMPONENTS WITH DISTINCT SLOPES OF THE LINES OF INSTANTANEOUS FREQUENCY CAN BE DONE WITH LIMITED SUCCESS

NUWC-NL Technical Report 10015
19 February 1992

Explicit Solution of Difference Equation for the Wavenumber Response of Fluid-Loaded Stiffened Plate

Albert H. Nuttall

ABSTRACT

A method for solving a difference equation encountered for excitation of a line-driven, fluid-loaded, infinite flat plate with nonperiodic attached rib stiffeners is presented. For the case of excitation by Q complex exponentials, a linear set of Q simultaneous equations must be solved for each wavenumber k. For the most general excitation, a linear integral equation must be solved at each k. A possible shortcut for obtaining solutions at certain shifted values of wavenumber k is pointed out.

TABLE OF CONTENTS

LIST OF SYMBOLS

L	periodic inter rib spacing
Δ	offset of one set of rib stiffeners to another set
Q	number of sets of attached rib stiffeners, (1)
D	rigidity of plate, (1)
$w(x)$	transverse plate displacement, (1)
m	mass per unit area of plate, (1)
ω	applied excitation frequency (radians/second), (1)
$P_e(x)$	external pressure due to line-force, (1)
$P_a(x,0)$	acoustic pressure on upper surface of plate, (1)
$P_q(x)$	total pressure exerted by q-th set of rib stiffeners
k	wavenumber, (2)
$W(k)$	wavenumber response, (2)
$F(k)$	auxiliary function, (2)
$Y(k)$	auxiliary function, (2)
k_L	wavenumber defined by periodic spacing, (2)
B_q	dynamic structural mass of q-th rib stiffener set, (2)
Δ_q	q-th offset distance, (2)
F_o	magnitude of applied line force, (3)
x_o	point of application of line force, (3)
$S(k)$	auxiliary function, (4)
k_b	free in-vacuum plate wavenumber, (4)
ρ_o	mass density of acoustic fluid, (4)
k_o	acoustic wavenumber ω/c_o, (4)
m'_q	mass per unit length of q-th set of rib stiffeners

m,n,p,q	integers
ε	real nonzero variable, (5)
$\{A_n\}$	given sequence, (5)
α_q	q-th exponential factor, (6)
$\underline{W}_q(k)$	auxiliary summation function, (8)
$\underline{F}_q(k)$	auxiliary summation function, (13)
$\underline{Y}_{pq}(k)$	auxiliary summation function, (14)
N(k)	numerator term for Q = 2, (20)
D(k)	denominator term for Q = 2, (21)
B(u)	Fourier sum, (22)
$\underline{W}(u,k)$	auxiliary summation function, (24)
$\underline{F}(u,k)$	auxiliary summation function, (29)
$\underline{Y}(v-u,k)$	auxiliary summation function, (30)
$\underline{W}_p(u,k)$	p-th order approximation to $\underline{W}(u,k)$, (33)

EXPLICIT SOLUTION OF DIFFERENCE EQUATION FOR THE WAVENUMBER RESPONSE OF FLUID-LOADED STIFFENED PLATE

INTRODUCTION*

Acoustic radiation from the upper surface of a fluid-loaded stiffened plate has been examined by Cray [1]. The formulation developed there produces an implicit algebraic equation for the Fourier transformed plate wavenumber response. This response must be manipulated into an explicit form in order to obtain a solution for the plate's structural response and to obtain the near- and far-field generated acoustic response. The following is a brief synopsis of the formulation for the acoustic radiation from a stiffened plate.

The infinite plate investigated in [1] was configured to have two infinite sets of attached rib stiffeners. The stiffeners composing a given set were identical and were spaced periodically with distance L. One set of stiffeners, however, was shifted by an amount Δ from the other set. In this manner, portions of the plate were configured with repeating sections having nonperiodic rib spacing.

The governing equation of motion for the surface displacement of the fluid-loaded isotropic plate for an applied line-force, with Q sets of attached rib stiffeners, is given by

$$D \frac{d^4 w(x)}{dx^4} - m\,\omega^2\, w(x) = P_e(x) - P_a(x,0) - \sum_{q=1}^{Q} P_q(x) \; , \qquad (1)$$

*The introduction was contributed by Dr. Benjamin A. Cray.

where D is the rigidity of the plate, $w(x)$ is the transverse plate displacement, m is the mass per unit area of the plate, ω is the applied excitation frequency, $P_e(x)$ is the external pressure due to the applied line-force, $P_a(x,0)$ is the acoustic pressure acting on the upper surface of the plate, and $\{P_q(x)\}$ are the total pressures exerted by each set of attached rib stiffeners.

Rotary inertia and shear deformation effects within the plate and rib stiffeners were neglected. The stiffeners exerted reactive forces upon the plate but applied no angular moments.

Equation (1) was transformed, term by term, using exponential Fourier transforms. The spatial transform variable, k, has the physical significance of wavenumber. It was assumed that the transforms are well defined and exist over the entire domain of integration.

Upon transforming (1) into the wavenumber domain, the implicit form of the wavenumber response, for the fluid-loaded stiffened plate, can be written as

$$W(k) = F(k) - Y(k) \sum_{n=-\infty}^{\infty} W(k + nk_L) \times$$

$$\times \left(B_1 + B_2 \exp\left(ink_L\Delta_1\right) + \cdot \cdot \cdot + B_Q \exp\left(ink_L\Delta_{Q-1}\right)\right) , \qquad (2)$$

where W(k) is the (transformed) wavenumber response. The quantities on the right-hand side of (2) are defined as

$$F(k) = \frac{F_o \exp(-ikx_o)}{S(k)} , \quad Y(k) = \frac{1}{S(k)} , \qquad (3)$$

where

$$S(k) = \begin{cases} D\left(k^4 - k_b^4\right) - \dfrac{i\ \rho_o\ \omega^2}{\sqrt{k_o^2 - k^2}} & \text{for } |k| < k_o \\ D\left(k^4 - k_b^4\right) - \dfrac{\rho_o\ \omega^2}{\sqrt{k^2 - k_o^2}} & \text{for } |k| > k_o \end{cases}, \tag{4}$$

where $k_b = (m\ \omega^2/D)^{1/4}$ is the free in-vacuum plate wavenumber, $k_L = 2\pi/L$ is the wavenumber defined by the periodic spacing, and $B_q = m'_q\ \omega^2/L$ is the dynamic structural mass of the q-th rib stiffener set. Also, Δ_q is the q-th offset distance.

Notice in (2) that the wavenumber or spectral response W(k) appears implicitly. The difficulty now lies in determining an explicit expression for the plate's wavenumber response W(k). That is, it is necessary to manipulate (2) such that the summation which contains the shifted wavenumber responses $\{W(k + nk_L)\}$ may be rewritten in terms of known quantities.

In [1], Cray obtained an explicit expression for W(k) for the case of a single offset, that is, Q = 2 in (2). This corresponds to two sets of rib stiffeners, where one set is offset from the other. The following development is a significant extension of the configuration of two sets of rib stiffeners to the case of an arbitrary number of rib stiffener sets, each with different offsets $\{\Delta_q\}$, and to arbitrary excitation functions.

DESCRIPTION AND SOLUTION OF DIFFERENCE EQUATION

In the following mathematical development, k and ε are arbitrary real variables, while m,n,p,q are integers. Also, summations without limits are from $-\infty$ to $+\infty$. We are interested in finding the solution $W(k)$ to the difference equation

$$W(k) = F(k) - Y(k) \sum_n A_n \, W(k + \varepsilon n) \, , \tag{5}$$

where $F(k)$ and $Y(k)$ are given known functions of k, and sequence $\{A_n\}$ is also known. For $\varepsilon = 0$, the solution tc (5) is immediate; hence, ε is nonzero and fixed in the following.

In fact, we are interested in the particular case where A_n is a finite sum of Q complex exponentials in n (compare with the second line of (2)):

$$A_n = \sum_{q=1}^{Q} B_q \exp(in\alpha_q) \quad \text{for all } n \, , \tag{6}$$

where $\{\alpha_q\}$, $1 \le q \le Q$, are arbitrary (complex) <u>distinct</u> constants, and $\{B_q\}$ are arbitrary complex constants. The $\{B_q\}$ and $\{\alpha_q\}$ are given. Furthermore, since

$$\exp(in\alpha_q) = \exp(in[\alpha_q - p2\pi]) \quad \text{for all } p \, , \tag{7}$$

there is <u>no</u> <u>loss</u> of generality in assuming that $|\mathrm{Re}(\alpha_q)| \le \pi$ for all $1 \le q \le Q$. If $\mathrm{Im}(\alpha_q)$ is nonzero for any q, then $|A_n|$ cannot remain bounded at both limits of $n = \pm\infty$. The earlier case solved in [1] corresponded to $Q = 2$ and the special value of $\alpha_1 = 0$.

DEFINITION AND PROPERTIES OF AUXILIARY FUNCTIONS

Define functions

$$\underline{W}_q(k) = \sum_n W(k + \varepsilon n) \exp(in\alpha_q) \quad \text{for } 1 \le q \le Q \,. \tag{8}$$

(The additional dependence of $\underline{W}_q(k)$ on ε is suppressed notationally.) It then follows that

$$\underline{W}_q(k + \varepsilon m) = \sum_n W(k + \varepsilon m + \varepsilon n) \exp(in\alpha_q) =$$

$$= \sum_p W(k + \varepsilon p) \exp(i(p-m)\alpha_q) = \exp(-im\alpha_q)\, \underline{W}_q(k) \,, \tag{9}$$

where we let $p = m + n$ and used (8). This is the key relation regarding the functions $\underline{W}_q(k)$ defined in (8), namely

$$\underline{W}_q(k + \varepsilon m) = \exp(-im\alpha_q)\, \underline{W}_q(k) \quad \text{for } 1 \le q \le Q \,. \tag{10}$$

With the aid of (6) and (8), the summation on n in (5) can now be manipulated into the form

$$\sum_n A_n W(k + \varepsilon n) = \sum_n W(k + \varepsilon n) \sum_{q=1}^{Q} B_q \exp(in\alpha_q) = \sum_{q=1}^{Q} B_q\, \underline{W}_q(k) \,. \tag{11}$$

This enables us to express (5) in the alternative form

$$W(k) = F(k) - Y(k) \sum_{q=1}^{Q} B_q\, \underline{W}_q(k) \,. \tag{12}$$

SOLUTION OF EQUATION (12)

At this point, in analogy to (8), it is convenient to define two additional functions, namely,

$$\underline{F}_q(k) = \sum_n F(k + \varepsilon n) \exp(in\alpha_q) \quad \text{for } 1 \le q \le Q , \tag{13}$$

$$\underline{Y}_{pq}(k) = \sum_n Y(k + \varepsilon n) \exp\left[in(\alpha_p - \alpha_q)\right] \quad \text{for } 1 \le p,q \le Q . \tag{14}$$

Now, replace k by k + εm in (12) and use (10), obtaining

$$W(k + \varepsilon m) = F(k + \varepsilon m) - Y(k + \varepsilon m) \sum_{q=1}^{Q} B_q \exp(-im\alpha_q) \underline{W}_q(k) . \tag{15}$$

When this equation is multiplied by $\exp(im\alpha_p)$ and summed over all m, there follows, by use of (8), (13), and (14),

$$\underline{W}_p(k) = \underline{F}_p(k) - \sum_{q=1}^{Q} \underline{Y}_{pq}(k) B_q \underline{W}_q(k) \quad \text{for } 1 \le p \le Q . \tag{16}$$

This relation constitutes Q linear equations (at each k) in the Q unknown functions $\underline{W}_q(k)$, $1 \le q \le Q$. The remaining quantities in (16) can be obtained from (6), (13), and (14).

When (16) is solved for all the $\{\underline{W}_q(k)\}$, then (12) directly yields the original quantity of interest, namely W(k) in (5), for that <u>particular</u> value of k. Although this procedure appears to require the solution of a new set of Q simultaneous linear equations for each k of interest, there is a shortcut that might be useful in some cases. Namely, reference to (15) reveals that

$W(k + \varepsilon m)$ can now be calculated easily for nonzero m, for <u>that</u> k, once $F(k + \varepsilon m)$ and $Y(k + \varepsilon m)$ have been computed, without the need for another solution set. The <u>same</u> solutions $\{\underline{W}_q(k)\}$ are used on the right-hand side of (15), regardless of the value of m under consideration; only the finite summation in (15) need be redone. Thus, the solution W to (5) at arguments k, $k \pm \varepsilon$, $k \pm 2\varepsilon, \ldots$ can be found from the solution of one set of Q simultaneous equations, (16).

For the special case of $Q = 1$, (16) can be immediately solved to yield [1]

$$\underline{W}_1(k) = \frac{\underline{F}_1(k)}{1 + B_1\, \underline{Y}_{11}(k)} = \frac{\sum_n F(k + \varepsilon n) \exp(in\alpha_1)}{1 + B_1 \sum_n Y(k + \varepsilon n)} . \tag{17}$$

Then (12) yields the explicit result (for $Q = 1$)

$$W(k) = F(k) - B_1\, Y(k) \frac{\sum_n F(k + \varepsilon n) \exp(in\alpha_1)}{1 + B_1 \sum_n Y(k + \varepsilon n)} . \tag{18}$$

In this form, it is possible to directly evaluate $W(k)$ at any k of interest merely by computing the terms encountered on the right-hand side of (18). However, shortcut (15) is still better for the particular arguments $\{k + \varepsilon m\}$.

For Q = 2, when the solutions of (16) are substituted into (12), the function W(k) is given by the explicit expression

$$W(k) = F(k) - Y(k)\,\frac{N(k)}{D(k)}\,, \tag{19}$$

where

$$N(k) = B_1\,\underline{F}_1(k) + B_2\,\underline{F}_2(k) + B_1\,B_2\,\times$$
$$\times\left[\underline{F}_1(k)\,\underline{Y}_{22}(k) + \underline{F}_2(k)\,\underline{Y}_{11}(k) - \underline{F}_1(k)\,\underline{Y}_{21}(k) - \underline{F}_2(k)\,\underline{Y}_{12}(k)\right] \tag{20}$$

and

$$D(k) = 1 + B_1\,\underline{Y}_{11}(k) + B_2\,\underline{Y}_{22}(k) +$$
$$+ B_1\,B_2\left[\underline{Y}_{11}(k)\,\underline{Y}_{22}(k) - \underline{Y}_{12}(k)\,\underline{Y}_{21}(k)\right]\,. \tag{21}$$

A program for the evaluation of (19) - (21), for the case where Y(k) is real, is presented in the appendix; also, B_1, B_2, α_1, and α_2 are real. Then D(k) is real, but N(k), F(k), and W(k) are complex. Also, $\underline{Y}_{22}(k) = \underline{Y}_{11}(k)$ is real, while $\underline{F}_1(k)$, $\underline{F}_2(k)$, and $\underline{Y}_{21}(k)^* = \underline{Y}_{12}(k)$ are complex. The program uses these properties.

In general, for any Q, if (16) is solved <u>analytically</u> for the set $\{\underline{W}_q(k)\}$ in terms of $\{B_q\}$, $\{\underline{F}_q(k)\}$, and $\{\underline{Y}_{pq}(k)\}$, these results can be substituted or utilized in (12) to get an explicit expression for W(k) that is valid for all k. For large Q, this will be impractical; instead, numerical solution of the Q simultaneous equations, (16), will be required at each k of interest. The only exception is to use (15) for the particular arguments $\{k + \varepsilon m\}$. (All results above actually hold for k and ε complex.)

SOLUTION FOR GENERAL SEQUENCE $\{A_n\}$

In this section, we no longer restrict sequence $\{A_n\}$ to have exponential form (6). Rather, for bounded given sequence $\{A_n\}$, define function

$$B(u) = \frac{1}{2\pi} \sum_n A_n \exp(-inu) \quad \text{for } |u| < \pi , \tag{22}$$

where u is real. Then, the sequence values can be found from

$$A_n = \int_{-\pi}^{\pi} du\, B(u) \exp(inu) \quad \text{for all } n . \tag{23}$$

The fastest variation with n that (23) allows for A_n is $\exp(\pm in\pi)$; however, this is no loss of generality, as seen by reference to (7). The case considered earlier in (6) corresponds to B(u) being composed of a set of Q impulses of area B_q located at $u = \alpha_q$, with $|\alpha_q| \le \pi$, when α_q is real.

Now, define function

$$\underline{W}(u,k) = \sum_n W(k + \varepsilon n) \exp(inu) \quad \text{for } |u| \le \pi . \tag{24}$$

(Again, the notational dependence on ε is suppressed.) The key relation that $\underline{W}(u,k)$ satisfies is

$$\underline{W}(u,k + \varepsilon m) = \sum_n W(k + \varepsilon m + \varepsilon n) \exp(inu) =$$

$$= \sum_p W(k + \varepsilon p) \exp(i(p-m)u) = \exp(-imu)\, \underline{W}(u,k) , \tag{25}$$

where we let $p = m + n$ and used (24). This relation,

$$\underline{W}(u,k + \varepsilon m) = \exp(-imu)\ \underline{W}(u,k) \ , \tag{26}$$

is the analogue of (10) earlier.

The summation in (5) can now be expressed as

$$\sum_n A_n\, W(k + \varepsilon n) = \sum_n W(k + \varepsilon n) \int_{-\pi}^{\pi} du\ B(u)\ \exp(inu) =$$

$$= \int_{-\pi}^{\pi} du\ B(u)\ \underline{W}(u,k) \ , \tag{27}$$

where we used (23) and (24). Therefore, the equation of interest, (5), now takes the form

$$W(k) = F(k) - Y(k) \int_{-\pi}^{\pi} du\ B(u)\ \underline{W}(u,k) \ . \tag{28}$$

We now define, in analogy to (13) and (14), the two functions

$$\underline{F}(u,k) = \sum_n F(k + \varepsilon n)\ \exp(inu) \quad \text{for } |u| \le \pi \ , \tag{29}$$

$$\underline{Y}(v-u,k) = \sum_n Y(k + \varepsilon n)\ \exp(in(v-u)) \quad \text{for } |u|,|v| \le \pi \ . \tag{30}$$

When we replace k by $k + \varepsilon m$ in (28), and use (26), there follows

$$W(k + \varepsilon m) = F(k + \varepsilon m) - Y(k + \varepsilon m) \int_{-\pi}^{\pi} du\ B(u)\ \underline{W}(u,k)\ \exp(-imu) \ . \tag{31}$$

We now multiply (31) by $\exp(imv)$, v real, and sum over all m, to

obtain

$$\underline{W}(v,k) = \underline{F}(v,k) - \int_{-\pi}^{\pi} du\ \underline{Y}(v-u,k)\ B(u)\ \underline{W}(u,k) \quad \text{for } |v| \le \pi\ , \tag{32}$$

where we used (24), (29), and (30). This is a linear integral equation, with kernel $\underline{Y}(v-u,k)\ B(u)$, for unknown $\underline{W}(u,k)$; that is, for each k of interest, a new linear integral equation must be solved. Then, $\underline{W}(u,k)$ will be known for $|u| \le \pi$ at that particular k.

Once $\underline{W}(u,k)$ is known, (28) gives desired solution W(k) at that particular k value, since F(k), Y(k), and B(u) are known functions. If a large number of k values are of interest, (32) can involve a great deal of computational effort; however, arbitrary bounded sequence $\{A_n\}$ is now allowed in (5). Furthermore, (31) then yields the solution for W at arguments k, $k \pm \varepsilon$, $k \pm 2\varepsilon$,... without the need to solve another integral equation.

The linear integral equation in (32) could be solved by recursion; for example, the p-th order approximation is given by

$$\underline{W}_p(v,k) \equiv \underline{F}(v,k) - \int_{-\pi}^{\pi} du\ \underline{Y}(v-u,k)\ B(u)\ \underline{W}_{p-1}(u,k) \quad \text{for } |v| \le \pi\ . \tag{33}$$

A possible starting value for this recursion is $\underline{W}_o(u,k) = \underline{F}(u,k)$.

SUMMARY

When the governing differential equation of motion for the surface displacement of the fluid-loaded isotropic plate for an applied line-force, with Q sets of attached rib stiffeners, is transformed into the wavenumber domain, a difference equation is encountered. Solution of this latter equation for exponential excitation, for a particular value of wavenumber k, is accomplished through the definition of auxiliary functions involving sums of displaced wavenumber responses with exponential factors. Finally, solution of a simultaneous set of Q linear equations completes the required calculations. For the most general excitation, the set of equations becomes infinite, that is, a linear integral equation must be solved.

APPENDIX. PROGRAM FOR (19) - (21)

In the following program for solution W(k) given by (19) - (21), specified function Y(k) is presumed real; however, F(k) is allowed to be complex. The following example corresponds to

$$F(k) = \exp(-.71\ k^2) + i\ \exp(-.91\ k^2 - k)\ ,$$

$$Y(k) = \exp(-.63\ k^2 + k)\ .$$

```
10    K=.7                    !  k, wavenumber
20    E=.4                    !  epsilon
30    L1=.1                   !  alpha1
40    L2=.2                   !  alpha2
50    B1=1.1
60    B2=1.3
70    CALL Fqk(K,E,L1,L2,F1r,F1i,F2r,F2i)       ! (13)
80    CALL Ypqk(K,E,L1,L2,Y11,Y12r,Y12i)        ! (14)
90    B=B1*B2
100   Y=Y11-Y12r
110   D=Y11*Y11-Y12r*Y12r-Y12i*Y12i
120   D=1.+(B1+B2)*Y11+B*D                      ! (21)
130   Nr=(F1r+F2r)*Y-(F1i-F2i)*Y12i
140   Nr=B1*F1r+B2*F2r+B*Nr                     ! (20)
150   Ni=(F1i+F2i)*Y+(F1r-F2r)*Y12i
160   Ni=B1*F1i+B2*F2i+B*Ni                     ! (20)
170   PRINT
180   PRINT "DENOMINATOR = ";D
190   Y=FNY(K)/D
200   CALL F(K,Fr,Fi)
210   Wr=Fr-Y*Nr
220   Wi=Fi-Y*Ni                                ! (19)
230   PRINT Wr,Wi
240   END
250   !
```

```
260     SUB Fqk(K,E,L1,L2,F1r,F1i,F2r,F2i)        ! (13)
270     DOUBLE N             !  INTEGER
280     F1r=F1i=F2r=F2i=0.
290     FOR N=-40 TO 40
300     CALL F(K+E*N,Fr,Fi)
310     IF ABS(N)=40 THEN PRINT ABS(Fr)+ABS(Fi);
320     C=COS(N*L1)
330     S=SIN(N*L1)
340     F1r=F1r+Fr*C-Fi*S
350     F1i=F1i+Fr*S+Fi*C
360     C=COS(N*L2)
370     S=SIN(N*L2)
380     F2r=F2r+Fr*C-Fi*S
390     F2i=F2i+Fr*S+Fi*C
400     NEXT N
410     SUBEND
420     !

430     SUB Ypqk(K,E,L1,L2,Y11,Y12r,Y12i)         ! (14)
440     DOUBLE N             !  INTEGER
450     Y11=Y12r=Y12i=0.
460     FOR N=-40 TO 40
470     T=FNY(K+E*N)
480     IF ABS(N)=40 THEN PRINT T;
490     A=N*(L1-L2)
500     Y11=Y11+T
510     Y12r=Y12r+T*COS(A)
520     Y12i=Y12i+T*SIN(A)
530     NEXT N
540     SUBEND
550     !
560     SUB F(K,Fr,Fi)                            ! F(k)
570     Fr=Fi=0.
580     A=.71*K*K
590     IF A>100. THEN 610
600     Fr=EXP(-A)
610     A=.91*K*K+K
620     IF A>100. THEN 640
630     Fi=EXP(-A)
640     SUBEND
650     !
660     DEF FNY(K)                                ! real Y(k)
670     A=.63*K*K-K
680     IF A>100. THEN RETURN 0.
690     RETURN EXP(-A)
700     FNEND
```

REFERENCES

[1] B. A. Cray, **Near-Field and Far-Field Sound Radiation from a Line-Driven Fluid-Loaded Infinite Flat Plate Having Periodic and Nonperiodic Attached Rib Stiffeners**, NUWC-NL Technical Report TR 10013, Naval Undersea Warfare Center, New London, CT, March 1992.

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Exact Detection Performance of Multiple-Pulse Frequency-Shift Signals in a Partially-Correlated Fading Medium with Generalized Noncentral Chi-Squared Statistics

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ABSTRACT

The transmitted signal consists of K pulses separated in time,frequency space so as to be nonoverlapping. In passing through the medium to the receiver, each signal pulse is subjected to fading. In particular, pulse pairs which are closely spaced in time,frequency space can fade in a highly dependent fashion, while those more widely separated can have relatively independent fading behavior; that is, the transmitted frequency-shift-keyed signal pulses undergo partially-correlated fading of a very general character that contains both deterministic components as well as random components. The amplitude-fading statistics are not limited to be Rayleigh.

Additive zero-mean Gaussian noise, which is stationary over the total signal transmission time and which has a flat spectrum over the total signal bandwidth, is present at the input to the receiver, in addition to the fluctuating signal pulses (when present).

Receiver processing consists of matched filtering of each of the K time-delayed, frequency-shifted pulse locations, followed by squared-envelope detection, sampling, summation, and comparison of this decision variable with a fixed threshold for a statement on signal presence or absence.

The characteristic function of the decision variable is derived in closed form for a very general model of partially-correlated fading that subsumes Rayleigh, Rician, and noncentral

Chi probability densities for the amplitude variate as special cases. This characteristic function depends on the number K of signal pulses, the number M of fading components, the deterministic received signal energy in each fading component, the average random received signal energy in each fading component, as well as the received noise spectral density level. Numerous special cases are pointed out and specific results are given in detail.

An efficient expansion for the exceedance distribution, for one of the cases, is listed and exercised for a representative numerical example. Comparisons with earlier approximations reveal them to have been pessimistic by several dB in ranges considered typical for practical applications. Also, the effects of correlated fading of the signal pulses are found to be not overly detrimental until the normalized covariance coefficient of adjacent pulses gets larger than approximately .5.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

FSK	frequency shift keying
t	time, (1)
$s(t)$	transmitted real signal waveform, (1)
$\underline{s}(t)$	transmitted signal complex envelope, (1)
f_o	carrier frequency, (1)
Re	real part, (1)
$\tilde{E}$	transmitted signal energy, (2)
r	amplitude scaling in transmission, (3)
θ	phase shift in transmission, (3)
t_d	time delay of received signal, (3)
f_d	frequency shift of received signal, (3)
overbar	ensemble average, (4)
$n(t)$	received noise waveform, (5)
$\underline{n}(t)$	received noise complex envelope, (5)
N_o	received noise one-sided spectral level, (5), (10)
A	local reference amplitude scaling, (6)
ϕ	local reference phase shift, (6)
$\alpha(t)$	complex envelope of received waveform, (7)
$\beta(t)$	complex envelope of local reference, (7)
γ	output of receiver processor, (8), figure 1
μ	parameter, $2\ r\ A\ \tilde{E}$, (8), (17)
n_r, n_i	real and imaginary noise components, (9)
$\delta(t)$	delta function, (10)
σ^2	parameter, $2\ N_o\ A^2\ \tilde{E}$, (12), (17)
Prob	exceedance probability, (14)

$f_c(\xi)$	conditional characteristic function of γ, (16)
$\chi(j)$	j-th conditional cumulant of γ, (19)
d	conditional deflection criterion, (21)
K	number of transmitted signal pulses, (22)
r_k	amplitude scaling of k-th signal pulse, (22)
θ_k	phase shift of k-th signal pulse, (22)
A_k	local reference scaling for k-th pulse, (22)
ϕ_k	local reference phase shift for k-th pulse, (22)
$\alpha_k(t)$	complex envelope of k-th received waveform, (22)
$\beta_k(t)$	complex envelope of k-th local reference, (22)
$\underline{s}_k(t)$	transmitted k-th signal pulse complex envelope, (22)
$\tilde{E}_k$	transmitted signal energy in k-th pulse, (23)
μ_k	k-th parameter, $2\, r_k\, A_k\, \tilde{E}_k$, (24)
σ_k^2	k-th parameter, $2\, N_o\, A_k^2\, \tilde{E}_k$, (24)
$f_c(\xi)$	conditional characteristic function of γ, (26), (27)
$\chi(j)$	j-th conditional cumulant of γ, (28)
d_K	conditional deflection of γ, (30)
$\overline{d_K}$	average deflection of output γ in figure 1, (32), (33)
A	common receiver weighting for K pulses, (35), (38)
$\tilde{E}$	common transmitted energy of K signal pulses, (36), (38)
S	sum of K power fading variates $\{r_k^2\}$, (37)
$f_S(\xi)$	characteristic function of random variable S, (41)
$f_\gamma(\xi)$	unconditional characteristic function of output γ, (42)
q_k	r_k^2, power scaling of k-th signal pulse, (43), (44)
t_k	time location of k-th FSK signal pulse, (44)
f_k	frequency location of k-th FSK signal pulse, (44)
$q(t,f)$	continuous fading process, time t and frequency f, (45)

M	number of fading components, (45)
$c_m(t,f)$	deterministic component of fading model, (45)
$g_m(t,f)$	random component of fading model, (45)
R_{mn}	covariance of g_m with g_n, (47)
τ,ν	delay and shift parameters in covariance R_{mn}, (47)
D_{km}	deterministic received signal energy in m-th component of k-th pulse, (50)
D_k	deterministic received signal energy in k-th pulse, (51)
E_{km}	average random received signal energy in m-th component of k-th pulse, (52)
E_k	average random received signal energy in k-th pulse,(53)
$\tilde{q}_k$	zero-mean component of q_k, (54)
$\tilde{R}_{kj}$	covariance between q_k and q_j, (55)
$\tilde{R}_{kk}$	variance of q_k, (56), (58)
ρ_{kj}	covariance coefficient between q_k and q_j, (56)
δ_{mn}	1 for m = n; 0 otherwise, (61)
$r^{(m)}$	relative power measure of m-th random fading component, (67)
X	M×1 Gaussian random column vector, (71)
E	mean vector, (72)
C,Cov(X)	covariance matrix of vector X, (72), (B-1)
Λ	eigenvalue matrix of C, (73)
Q	normalized modal matrix of C, (73)
diag	diagonal matrix, (74)
λ_m	m-th eigenvalue of matrix Λ, $1 \le m \le M$, (74)
V_m	m-th eigenvector of matrix Q, (74)
v_{mn}	n-th component of vector V_m, (74)
ε_m	deterministic parameter, (75)

$f_q(\xi)$	characteristic function of power-scaling q(t,f), (76)
$\chi_q(p)$	p-th cumulant of q(t,f), (77)
$\underline{E}$	transformed mean vector, (91)
ε_n	components of $\underline{E}$, (91)
$S^{(m)}$	power sum over signal pulse numbers, $1 \le m \le M$, (95)
$f_S(\xi)$	characteristic function of sum S, (96)
$f^{(m)}(\xi)$	characteristic function of $S^{(m)}$, (97)
$Y^{(m)}$	K×1 random column vector, (98)
$C^{(m)}$	covariance matrix of vector $Y^{(m)}$, (100)
$\Lambda^{(m)}$	eigenvalue matrix of $C^{(m)}$, (101)
$Q^{(m)}$	normalized modal matrix of $C^{(m)}$, (101)
$\lambda_k^{(m)}$	k-th eigenvalue of matrix $\Lambda^{(m)}$, (102)
$V_k^{(m)}$	k-th eigenvector of matrix $Q^{(m)}$, (102)
$v_{kj}^{(m)}$	j-th *component of vector* $V_k^{(m)}$, (102)
$\varepsilon_k^{(m)}$	transformed mean vector, (103)
h_k	auxiliary constants, $1 \le k \le K$, (111)
$\chi_S(p)$	p-th cumulant of sum S, (122)
$\chi_\gamma(p)$	p-th cumulant of output γ, (125)
$\underline{C}^{(m)}$	normalized covariance matrix, (127)
$\underline{\Lambda}^{(m)}$	eigenvalue matrix of $\underline{C}^{(m)}$, (128)
$\underline{\lambda}_k^{(m)}$	k-th eigenvalue of matrix $\underline{\Lambda}^{(m)}$, (128)
$\underline{\varepsilon}_k^{(m)}$	auxiliary constants, (137)
$\underline{C}$	common normalized covariance matrix, (140)
$\underline{\Lambda}$	eigenvalue matrix of $\underline{C}$, (141)
$\underline{\lambda}_k$	k-th eigenvalue of matrix $\underline{\Lambda}$, (141)
$\underline{h}_k$	auxiliary constants, (145)
C	covariance matrix of $g_1(t,f)$, (152)

ϕ	normalized power-scaling, (161)
σ_m^2	variance of $g_m(t,f)$, (162)
f_ϕ	characteristic function of ϕ, (163)
η_m	normalized constants, (164)
PDF	probability density function, (166), figure 2
p_ϕ	probability density function of ϕ, (166)
θ	normalized amplitude-scaling, (167)
p_θ	probability density function of θ, (168)
C,S	auxiliary functions, (172)
η	constant, (174), (179)
I_N	modified Bessel function of order N, (180)
p_γ	probability density function of processor output γ
T_{mk}	auxiliary parameters, (182)
e_{mk}	auxiliary parameters, (182)
Prob	probability, (183)
u	threshold, (183)
g_p	expansion coefficient, (183), (186)
F	scale factor, (184)
α_p	auxiliary parameters, (185)
$H_n(x)$	auxiliary function, (187)
ψ_m	fractional strengths of random components, (190)
SNR	signal-to-noise ratio
Cov_1	normalized covariance $\dfrac{R_{11}(t_{k+1}-t_k,0)}{R_{11}(0,0)}$, below (193)
$Cov(\tau,\nu)$	normalized covariance $R_{11}(\tau,\nu)/R_{11}(0,0)$, (194)
det	determinant of matrix, (B-4)

Exact Detection Performance of Multiple-Pulse Frequency-Shift Signals in a Partially-Correlated Fading Medium with Generalized Noncentral Chi-Squared Statistics

INTRODUCTION

The transmission of a succession of time-delayed and/or frequency-shifted signal pulses through a fading medium leads to received waveforms which vary in amplitude and phase in a random fashion and with possibly complicated statistical dependencies. The evaluation of the detection capability of matched filter processing and incoherent combination in such fading situations and noise has been the subject of many studies over the years [1 - 15]. Some of these efforts have been aimed at obtaining approximations to the performance of the systems of interest, while others have yielded exact results in selected special cases.

Results for partially-correlated fading have been obtained in [3;4;6;7;9;10;11;12;14;15]. In particular, in [4;6;10;11;12;14], the characteristic functions of the decision variables have been obtained in closed form for the case of partially correlated fading of the received signal pulses with Rayleigh amplitude statistics and additive Gaussian noise. The results in [12] are approximate; however, the signal amplitude fading is not limited to Rayleigh statistics there, but rather can have a Chi distribution of an arbitrary number of degrees of freedom.

The case of independent signal fading in noise, but allowing correlated clutter, was addressed in [13]. An extension of [12]

to partially correlated signal fading in a system with normalization was solved in an approximate fashion in [14]. Finally, a case of partially correlated signal fading with Rayleigh amplitude statistics in the presence of noise and K-distributed clutter was solved exactly in [15].

Here, we will give exact results for a fading medium which has both deterministic components as well as random components with arbitrary covariance coefficients and additive Gaussian noise. Also, frequency shift keyed (FSK) signals will be allowed, with arbitrary (non-overlapping) occupancy in time, frequency space. In addition, the signal amplitude statistics will not be limited to Rayleigh, but will include the range of possibilities inherent in the noncentral Chi distribution of an arbitrary number of degrees of freedom. The end result is a closed form for the characteristic function of the decision variable that, although complicated in appearance, is amenable to efficient computer evaluation of the detection characteristics by means of a single fast Fourier transform (FFT). No matrix inverses are required. However, a novel expansion technique for the detection probability is derived which is efficient and accurate; a program incorporating this expansion is listed and exercised for several examples.

Extensions to a more general form of processor will also be solved but not evaluated. In particular, the characteristic function of the most general complex second-order form of arbitrarily-correlated complex Gaussian random variables with arbitrary means will be obtained in closed form.

RECEIVER PROCESSING FOR A SINGLE PULSE

In this section, we will describe the basic model of the single-pulse transmitted signal, the received signal with fading, the received noise, and the receiver processing. The extension to multiple signal pulses will be undertaken in the next section.

We presume that the transmitted real signal s(t) is narrowband with low-frequency complex envelope $\underline{s}(t)$ superposed on carrier frequency f_o:

$$s(t) = \mathrm{Re}\{\underline{s}(t)\ \exp(i2\pi f_o t)\}\ . \tag{1}$$

The transmitted signal energy is then

$$\tilde{E} \equiv \int dt\ [s(t)]^2 = \frac{1}{2}\int dt\ |\underline{s}(t)|^2\ , \tag{2}$$

where integrals without limits are over $(-\infty,+\infty)$. The time-bandwidth product of the single pulse complex envelope $\underline{s}(t)$ is arbitrary; thus, for example, $\underline{s}(t)$ could contain linear frequency modulation along with rectangular or Gaussian amplitude modulation.

The received signal waveform is

$$\mathrm{Re}\{r\ \underline{s}(t-t_d)\ \exp[i2\pi(f_o+f_d)t + i\theta]\}\ , \tag{3}$$

where r is a (dimensionless) amplitude scaling, t_d is a time delay, f_d is a frequency shift, and θ is a phase shift. Real random unknowns r and θ do not vary with time over the duration of pulsed signal $\underline{s}(t)$. Delay t_d and shift f_d are presumed known at the receiver. The average received signal energy is, using

(3) and (2),

$$\overline{r^2}\ \frac{1}{2} \int dt\ |\underline{s}(t)|^2 = \overline{r^2}\ \tilde{E}\ . \tag{4}$$

The received zero-mean additive noise waveform n(t) is

$$n(t) = \mathrm{Re}\{\underline{n}(t)\ \exp(i2\pi f_o t)\}\ , \tag{5}$$

which is presumed stationary over the duration of the signal. The spectrum of received noise n(t) in the neighborhood of frequency f_o is flat, with one-sided spectral level N_o watts/Hz.

The reference waveform employed at the receiver corresponds to the matched filter to the transmitted signal, namely

$$A\ \mathrm{Re}\{\underline{s}(t-t_d)\ \exp[i2\pi(f_o+f_d)t + i\phi]\}\ , \tag{6}$$

which utilizes knowledge of t_d and f_d. The local reference level A and local phase shift ϕ in (6) are irrelevant to the processing employed here; that is, the performance in terms of the receiver operating characteristics (detection probability P_D versus false alarm probability P_F) is independent of A and ϕ, at least for this case of one signal pulse.

We now define the two analytic functions

$$\alpha(t) = r\ \underline{s}(t-t_d)\ \exp[i2\pi(f_o+f_d)t+i\theta] + \underline{n}(t)\ \exp(i2\pi f_o t)\ ,$$

$$\beta(t) = A\ \underline{s}(t-t_d)\ \exp[i2\pi(f_o+f_d)t+i\phi]\ , \tag{7}$$

which are recognized as corresponding to the received waveform and the local reference, respectively. The total output of the matched-filter squared-envelope detector to the received signal and noise waveform is then given by

$$\gamma = \left| \int dt\ \alpha(t)\ \beta^*(t) \right|^2 =$$

$$= \left| r \exp(i\theta)\ A \int dt\ |\underline{s}(t-t_d)|^2 + A \int dt\ \underline{n}(t)\ \underline{s}^*(t-t_d)\ \exp(-i2\pi f_d t) \right|^2$$

$$= \left| 2\ r\ A\ \tilde{E} + n_r + i\ n_i \right|^2 \equiv \left| \mu + n_r + i\ n_i \right|^2 , \qquad (8)$$

where we used (2) and defined the zero-mean processor output complex noise variate

$$n_r + i\ n_i = \exp(-i\theta)\ A \int dt\ \underline{n}(t)\ \underline{s}^*(t-t_d)\ \exp(-i2\pi f_d t) . \qquad (9)$$

The <u>two</u> covariances of the received noise complex envelope $\underline{n}(t)$ are derived in appendix A; they are given by (A-8) and (A-11) as

$$\overline{\underline{n}(t_1)\ \underline{n}(t_2)} = 0 , \quad \overline{\underline{n}(t_1)\ \underline{n}^*(t_2)} = 2\ N_o\ \delta(t_1-t_2) . \qquad (10)$$

Use of these relations with (9) enables us to determine the two averages

$$\overline{(n_r + i\ n_i)^2} = \overline{n_r^2} - \overline{n_i^2} + i\ 2\ \overline{n_r\ n_i} = 0 ,$$

$$\overline{|n_r + i\ n_i|^2} = \overline{n_r^2} + \overline{n_i^2} = 2\ N_o\ A^2 \int dt\ |\underline{s}(t-t_d)|^2 = 4\ N_o\ A^2\ \tilde{E} , \qquad (11)$$

where we also used (2). Combining the information in the two lines of (11), there follows, for the two zero-mean real variates n_r and n_i,

$$\overline{n_r^2} = \overline{n_i^2} = 2\ N_o\ A^2\ \tilde{E} \equiv \sigma^2 , \quad \overline{n_r\ n_i} = 0 . \qquad (12)$$

These results are independent of the parameters θ, t_d, and f_d.

Since noise input $\underline{n}(t)$ is Gaussian and operation (9) is linear, real variates n_r and n_i are Gaussian with joint probability density function

$$p(n_r, n_i) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_r^2 + n_i^2}{2\sigma^2}\right) . \tag{13}$$

From this result, we can derive the conditional exceedance distribution function of matched filter output γ in (8), for a given value of random variable r. In terms of the two auxiliary parameters μ and σ^2 defined in (8) and (12) respectively, we have, for u > 0, exceedance probability

$$\mathrm{Prob}(\gamma > u) = \mathrm{Prob}\left((\mu + n_r)^2 + n_i^2 > u\right) =$$

$$= \iint\limits_C dx\ dy\ \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2 + y^2}{2\sigma^2}\right) =$$

$$= \frac{1}{2\pi\sigma^2} \int_{\sqrt{u}}^{\infty} dr'\ r' \int_{-\pi}^{\pi} d\theta'\ \exp\left(-\frac{r'^2 - 2\mu r'\cos\theta' + \mu^2}{2\sigma^2}\right) =$$

$$= \frac{1}{\sigma^2} \int_{\sqrt{u}}^{\infty} dr'\ r'\ \exp\left(-\frac{r'^2 + \mu^2}{2\sigma^2}\right) I_o\left(\frac{\mu r'}{\sigma^2}\right) = Q\left(\frac{\mu}{\sigma}, \frac{\sqrt{u}}{\sigma}\right) , \tag{14}$$

where C is the region exterior to a circle of radius $\sqrt{u}$ centered at the origin of the x,y plane, and Q is Marcum's function [1].

The conditional probability density function of γ, for the same given value of amplitude scaling r, is

$$p(u) = -\frac{d}{du}\mathrm{Prob}(\gamma > u) = \frac{1}{2\sigma^2} \exp\left(-\frac{u + \mu^2}{2\sigma^2}\right) I_o\left(\frac{\mu\sqrt{u}}{\sigma^2}\right) , \tag{15}$$

for $u > 0$. Therefore, the conditional characteristic function of the squared-envelope detector output γ, for given r, is

$$f_c(\xi) = \int du \exp(i\xi u)\ p(u) = \frac{1}{2\sigma^2} \int_0^\infty du \exp\left(i\xi u - \frac{u + \mu^2}{2\sigma^2}\right) I_o\left(\frac{\mu \sqrt{u}}{\sigma^2}\right)$$

$$= \frac{1}{1 - i\xi 2\sigma^2} \exp\left(\frac{i\xi\mu^2}{1 - i\xi 2\sigma^2}\right) , \qquad (16)$$

where we used [16; 6.631 4]. The two important parameters here were defined in (8) and (12) according to

$$\mu = 2\ r\ A\ \tilde{E}\ , \quad \sigma^2 = 2\ N_o\ A^2\ \tilde{E}\ . \qquad (17)$$

Notice that, in this case of matched filtering, the detailed behavior of complex envelope $\underline{s}(t)$ is not relevant; only the total transmitted signal energy $\tilde{E}$ is of consequence to the conditional characteristic function (16) of output γ in (8). The results in (16) and (17) are independent of the particular value of the medium phase shift θ encountered in (3) or (7); this is due to the processing method adopted at the receiver, namely envelope detection of the matched filter output.

The cumulants of random variable γ, for given fixed amplitude scaling r, are available by developing the logarithm of (16) in a power series in $i\xi$:

$$\ln f_c(\xi) = -\ln(1 - i\xi 2\sigma^2) + i\xi\mu^2 \left(1 - i\xi 2\sigma^2\right)^{-1} =$$

$$= \sum_{j=1}^{\infty} (i\xi)^j \left(\frac{(2\sigma^2)^j}{j} + \mu^2 (2\sigma^2)^{j-1}\right) . \qquad (18)$$

There follows immediately the j-th (conditional) cumulant of γ as

$$\chi(j) = (j-1)!\ (2\sigma^2)^j \left(1 + j\,\frac{\mu^2}{2\sigma^2}\right) \quad \text{for } j \geq 1\ . \tag{19}$$

In particular, the first two cumulants are

$$\chi(1) = 2\sigma^2 + \mu^2\ , \quad \chi(2) = 4\sigma^4 \left(1 + \frac{\mu^2}{\sigma^2}\right)\ . \tag{20}$$

It is convenient to define a conditional deflection criterion at the squared-envelope detector output γ as the ratio of the difference of means, with and without signal, to the noise-alone standard deviation. That is,

$$d \equiv \frac{\chi_{S+N}(1) - \chi_N(1)}{\chi_N^{1/2}(2)} = \frac{\mu^2}{2\sigma^2} = r^2\,\frac{\tilde{E}}{N_o}\ , \tag{21}$$

where we used (20) and (17). This conditional deflection depends on amplitude scaling value r, of course. However, it is independent of local reference values A and ϕ in (6), as expected, since performance measures should not depend on receiver processor absolute levels or phase shifts, at least for the case of a single pulse.

RECEIVER PROCESSING FOR MULTIPLE SIGNAL PULSES

In this section we will generalize to the case where K signal pulses are transmitted, all of which are nonoverlapping in time, frequency space; the results derived in the previous section will be used freely in the sequel. The transmitted energy in the k-th pulse is $\tilde{E}_k$, where $1 \le k \le K$. The k-th signal pulse undergoes (dimensionless) amplitude scaling r_k and phase shift θ_k in transmission through the medium to the receiver. For each k, random variables r_k and θ_k do not vary with time over the duration of the k-th individual signal pulse; this is a generalization of (3). The k-th local reference waveform employed at the receiver utilizes constant amplitude scaling A_k and phase shift ϕ_k, which can be chosen for best performance; compare (6).

It is now necessary to generalize the definitions in (7) to K pairs of analytic functions, namely

$$\alpha_k(t) = r_k \, \underline{s}_k(t-t_d) \exp[i2\pi(f_o+f_d)t+i\theta_k] + \underline{n}(t) \exp(i2\pi f_o t) \; ,$$

$$\beta_k(t) = A_k \, \underline{s}_k(t-t_d) \exp[i2\pi(f_o+f_d)t+i\phi_k] \; , \qquad (22)$$

which correspond to the k-th received waveform and local reference, respectively. A block diagram of the signal processing employed at the receiver is depicted in figure 1. The k-th signal complex envelope $\underline{s}_k(t)$ in (22) includes the time delay t_k and the frequency shift f_k associated with the k-th pulse, in accordance with the FSK pattern employed at the transmitter and known to the receiver. The K matched filter outputs are envelope detected, squared, and then summed to yield

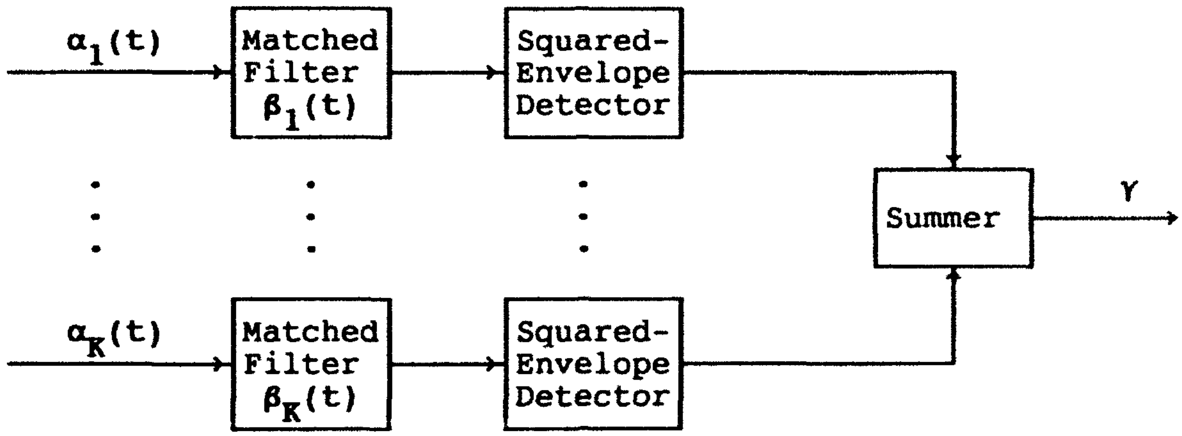

Figure 1. Block Diagram of Receiver Processing

output γ, which is compared with a threshold for purposes of declaring signal presence or absence. Observe that the matched filter $\beta_k(t)$ incorporates the local reference scaling A_k and phase shift ϕ_k and therefore accounts for (complex) weighting prior to the summation indicated in figure 1. The absolute level of weights $\{A_k\}$ does not matter and can be chosen freely; however, their values <u>relative</u> to each other will affect the detection capability of the processor in figure 1.

The energy in the k-th (real) reference waveform is, from (22) and (2),

$$\frac{1}{2} \int dt \, |\beta_k(t)|^2 = A_k^2 \, \tilde{E}_k \, , \tag{23}$$

where $\tilde{E}_k$ is the transmitted energy in the k-th signal pulse. In analogy to (17), we define the K parameters

$$\mu_k = 2 \, r_k \, A_k \, \tilde{E}_k \, , \quad \sigma_k^2 = 2 \, N_o \, A_k^2 \, \tilde{E}_k \quad \text{for } 1 \le k \le K \, . \tag{24}$$

The noise density level N_o is a common constant applicable to all K pulses because the received noise is stationary over the total time extent, and it is white over the entire frequency band of the K received signals.

The K signals $\{\underline{s}_k(t)\}$ are nonoverlapping in time,frequency space, (t,f). Also, the received noise is Gaussian. Therefore, the K output noise variables from the matched filters in figure 1 are statistically independent of each other; mathematically, we are using orthogonality relation

$$\int dt\ \underline{s}_k(t)\ \underline{s}_j^*(t) = 0 \quad \text{for } k \neq j\ . \tag{25}$$

This is due to nonoverlapping, in time or frequency or both, of all the component FSK pulses $\{\underline{s}_k(t)\}$, each with their individual delay,shift t_k, f_k. These observations, plus relations (8) and (16), allow us to determine the conditional characteristic function of output γ from figure 1 in the compact form

$$f_c(\xi) = \prod_{k=1}^{K}\left(1 - i\xi\ 2\sigma_k^2\right)^{-1} \exp\left[i\xi \sum_{k=1}^{K} \frac{\mu_k^2}{1 - i\xi\ 2\sigma_k^2}\right] . \tag{26}$$

It should be noticed that this result is independent of the set of phase shifts $\{\theta_k\}$ encountered in transmission; see (22) and (24). The reason for this development is the method of receiver processing adopted in figure 1, namely envelope detection of each of the matched filter outputs prior to summation. However, (26) depends significantly on the amplitude scalings $\{r_k\}$ through parameters $\{\mu_k\}$ in (24); explicitly, the alternative form of (26) is

$$f_c(\xi) = \prod_{k=1}^{K}\left(1 - i\xi\, 4\, N_o\, A_k^2\, \tilde{E}_k\right)^{-1} \exp\left[i\xi \sum_{k=1}^{K} \frac{4\, r_k^2\, A_k^2\, \tilde{E}_k^2}{1 - i\xi\, 4\, N_o\, A_k^2\, \tilde{E}_k}\right] . \tag{27}$$

Also, characteristic function (27) is independent of the receiver set of phase shifts $\{\phi_k\}$ used in local references (22), again due to the method of processing employed in figure 1. However, (27) depends upon the receiver weights $\{A_k\}$ in a complicated fashion. In order to determine a reasonable method of selecting this set of weights, we consider the following development.

The (conditional) cumulants of system output random variable γ, for a given fixed set of amplitude scalings $\{r_k\}$, are available by developing the logarithm of (26) or (27) in a power series in $i\xi$. Reference to (18) and (19) immediately yields the j-th (conditional) cumulant of γ as

$$\chi(j) = (j-1)! \sum_{k=1}^{K} \left(2\sigma_k^2\right)^j \left(1 + j\, \frac{\mu_k^2}{2\sigma_k^2}\right) \quad \text{for } j \geq 1 . \tag{28}$$

In particular, the first two cumulants of γ are

$$\chi(1) = \sum_{k=1}^{K} \left(2\sigma_k^2 + \mu_k^2\right) , \quad \chi(2) = 4 \sum_{k=1}^{K} \left(\sigma_k^4 + \sigma_k^2\, \mu_k^2\right) . \tag{29}$$

We can now define a (conditional) deflection criterion d_K for the output of the processor in figure 1, exactly as was done in (21) earlier for the single pulse case. Substitution of (29) and (24) then yields

$$d_K = \frac{\sum_{k=1}^{K} \mu_k^2}{2\left(\sum_{k=1}^{K} \sigma_k^4\right)^{\frac{1}{2}}} = \frac{\sum_{k=1}^{K} r_k^2 \, A_k^2 \, \tilde{E}_k^2}{N_o \left(\sum_{k=1}^{K} A_k^4 \, \tilde{E}_k^2\right)^{\frac{1}{2}}} \ . \tag{30}$$

The absolute level of receiver weights $\{A_k\}$ cancels out of (30); however, the <u>relative</u> sizes of these weights does affect the value of deflection d_K attainable.

If the average <u>power</u> scaling of the k-th pulse, $\overline{r_k^2}$, is the same for all k, that is,

$$\overline{r_k^2} = \overline{r_1^2} \quad \text{for } 2 \le k \le K \ , \tag{31}$$

then the <u>average</u> deflection of output γ in figure 1 is, from (30),

$$\overline{d_K} = \frac{\overline{r_1^2}}{N_o} \frac{\sum_{k=1}^{K} A_k^2 \, \tilde{E}_k^2}{\left(\sum_{k=1}^{K} A_k^4 \, \tilde{E}_k^2\right)^{\frac{1}{2}}} \le \frac{\overline{r_1^2}}{N_o} \left(\sum_{k=1}^{K} \tilde{E}_k^2\right)^{\frac{1}{2}} \ , \tag{32}$$

with equality only if all the weights A_k are equal. This result is not surprising, since there is already a built-in dependence of matched filter $\beta_k(t)$ on the transmitted signal energy $\tilde{E}_k$; namely, (23) gives this energy as $A_k^2 \, \tilde{E}_k$. Thus, we have the conclusion that the "best" set of weights $\{A_k\}$, for the case of equal power scalings (31) in the medium, is uniform when local reference $\beta_k(t)$ is chosen according to (22).

More generally, the average deflection follows from (30) as

$$\overline{d_K} = \frac{\sum_{k=1}^{K} \overline{r_k^2}\, A_k^2\, \tilde{E}_k^2}{N_o \left(\sum_{k=1}^{K} A_k^4\, \tilde{E}_k^2\right)^{\frac{1}{2}}} \le \frac{1}{N_o} \left(\sum_{k=1}^{K} \overline{r_k^2}^{\,2}\, \tilde{E}_k^2\right)^{\frac{1}{2}} , \tag{33}$$

with equality if and only if the receiver weights satisfy

$$A_k = A\, \overline{r_k^2}^{\,\frac{1}{2}} \quad \text{for } 1 \le k \le K . \tag{34}$$

Scale factor A is arbitrary, reflecting the fact that the absolute level of set $\{A_k\}$ is irrelevant. It should be noted that the "best" receiver weights in (34) are independent of the transmitted signal energies $\{\tilde{E}_k\}$. This is again due to the fact that matched filter $\beta_k(t)$ in figure 1 and (22) already has an energy proportional to $\tilde{E}_k$; see (23).

Result (34) for the weights $\{A_k\}$ requires knowledge of the average power scaling $\overline{r_k^2}$ applied to the k-th pulse in transmission through the medium. But these average power scalings may be unknown or they may be independent of k, the particular pulse number. In this case, a reasonable choice of receiver weights is simply to take $A_k = A$ for $1 \le k \le K$. Then the conditional characteristic function in (27) of processor output γ reduces to

$$f_c(\xi) = \prod_{k=1}^{K} \left(1 - i\xi\, 4\, N_o\, A^2\, \tilde{E}_k\right)^{-1} \exp\left[i\xi 4A^2 \sum_{k=1}^{K} \frac{r_k^2\, \tilde{E}_k^2}{1 - i\xi\, 4\, N_o\, A^2\, \tilde{E}_k}\right] \tag{35}$$

In the (usual) case where all the transmitted signal energies in the K pulses are equal, that is, $\tilde{E}_k = \tilde{E}$ for $1 \le k \le K$, then

the conditional characteristic function in (35) reduces to

$$f_c(\xi) = \left(1 - i\xi\ 4\ N_o\ A^2\ \tilde{E}\right)^{-K} \exp\left[\frac{i\xi\ 4\ A^2\ \tilde{E}^2}{1 - i\xi\ 4\ N_o\ A^2\ \tilde{E}} \sum_{k=1}^{K} r_k^2\right] . \quad (36)$$

It is now very important to notice that the <u>only</u> way that the amplitude scaling factors $\{r_k\}$ enter this characteristic function is through the single quantity (sufficient statistic)

$$S = \sum_{k=1}^{K} r_k^2 , \quad (37)$$

which is the sum of all the <u>power</u>-fading variates r_k^2 on the K transmitted pulses. This simplified result in (36) holds when the following two reasonable conditions are satisfied:

$$\tilde{E}_k = \tilde{E} , \quad A_k = A \quad \text{for } 1 \le k \le K . \quad (38)$$

That is, the transmitted signal pulse energies are all equal and the receiver scalings $\{A_k\}$ in (22) are all taken equal. This particular case will receive most of our attention.

Since the receiver absolute scaling A in (38) is under our control and does not affect detectability performance, we take, for notational convenience and without loss of generality,

$$A^2 = \frac{1}{2\ N_o\ \tilde{E}} . \quad (39)$$

Then the conditional characteristic function in (36) of processor output γ in figure 1 simplifies to

$$f_c(\xi) = (1 - i2\xi)^{-K} \exp\left[\frac{i\xi}{1 - i2\xi} \frac{2\tilde{E}}{N_o} S\right] , \quad (40)$$

where we used (37).

Another important observation must be made at this point. Suppose the characteristic function of random variable S, defined by (37), is known; that is,

$$f_S(\xi) = \overline{\exp(i\xi S)} \tag{41}$$

is known. Then the unconditional characteristic function $f_\gamma(\xi)$ of processor output γ in figure 1 is obtained by averaging (40) over S and using (41); there follows

$$f_\gamma(\xi) = (1 - i2\xi)^{-K} f_S\left(\frac{\xi}{1 - i2\xi} \frac{2\tilde{E}}{N_o}\right) . \tag{42}$$

This compact form for the unconditional characteristic function of processor output γ depends critically upon being able to obtain the characteristic function $f_S(\xi)$ in (41) of the power-fading summation S defined in (37). Therefore, effort can be concentrated on attempting to determine (41), either exactly or by use of several low-order moments of S; see [17] for example. Of course, in the process, the amount of partial correlation between individual pairs of pulse power-fading variates $\{r_k^2\}$ in sum S will come into consideration. In any event, the characteristic function of sum S in (37) is the major item of interest; given this quantity, the processor output γ in figure 1 is completely characterized in terms of its characteristic function (42), at least when conditions (38) are satisfied.

The more general conditional characteristic function in (27), with arbitrary $\{\tilde{E}_k\}$ and $\{A_k\}$, will be treated later, after we have introduced a detailed model of the amplitude fadings $\{r_k\}$.

CHARACTERIZATION OF FADING

We are interested in obtaining characteristic function $f_S(\xi)$ of sum S of power-fading variates $\{r_k^2\}$ in (37); its use in (42) will then yield the desired characteristic function $f_\gamma(\xi)$ of receiver output γ in figure 1. However, we must first concentrate on characterizing the fading which gives rise to sum S.

COVARIANCE COEFFICIENTS OF POWER-FADING VARIATES $\{q_k\}$

Define the k-th power-fading variate q_k as the square of the amplitude-fading variate r_k in (22):

$$q_k = r_k^2 \quad \text{for } 1 \le k \le K \, . \tag{43}$$

The model of fading that we consider here is that power scaling q_k is a sample of a continuous fading process q(t,f), namely

$$q_k = q(t_k, f_k) \quad \text{for } 1 \le k \le K \, , \tag{44}$$

where two-dimensional function of time t and frequency f,

$$q(t,f) = \sum_{m=1}^{M} \left[c_m(t,f) + g_m(t,f) \right]^2 \, . \tag{45}$$

Sampling time t_k and frequency f_k correspond to the time-delay location and frequency-shift location, respectively, of the k-th FSK complex envelope signal $\underline{s}_k(t)$ employed in (22).

The number of fading components in model (45) is M. Each component contains a deterministic part $c_m(t,f)$ and a stationary zero-mean Gaussian part $g_m(t,f)$. Without loss of generality, the

nonrandom part satisfies

$$c_m(t,f) \geq 0 \quad \text{for all } m,\ t,\ f\ . \tag{46}$$

Random parts $\{g_m(t,f)\}$ of the joint Gaussian processes are stationary in both t and f, with covariances

$$\overline{g_m(t,f)\ g_n(t-\tau,f-\nu)} = R_{mn}(\tau,\nu) \quad \text{for } 1 \leq m,\ n \leq M\ . \tag{47}$$

(It is possible to generalize to nonstationary Gaussian processes $\{g_m(t,f)\}$; however, each covariance R_{mn} would then be a function of the four variables τ,ν,t,f instead of just differences τ,ν.) From (47), we have property

$$R_{nm}(-\tau,-\nu) = R_{mn}(\tau,\nu)\ . \tag{48}$$

Fading model (44) - (45) does not constitute a multipath medium but does mimic its net effect. Every transmitted pulse $\underline{s}_k(t)$ undergoes just one common time delay t_d and frequency shift f_d, as indicated in (22), which are known and utilized by the receiver. Rather, each signal pulse simply undergoes a different phase shift θ_k and amplitude scaling r_k in (22), the latter of which is characterized through power scaling q_k in (43) - (45).

The latter random variable, q_k, is more general than non-central Chi-squared because the random components $\{g_m(t,f)\}$ in (45) can have unequal variances $\{R_{mm}(0,0)\}$ and can be correlated with each other; that is, we allow $R_{mn}(\tau,\nu) \neq 0$ for $m \neq n$.

The mean of power-fading variate q_k is given by

$$\overline{q_k} = \sum_{m=1}^{M} \left[c_m^2(t_k,f_k) + R_{mm}(0,0)\right] \quad \text{for } 1 \leq k \leq K\ , \tag{49}$$

where we used (44), (45), and (47). This quantity is independent of pulse number k if all M deterministic components are zero, or if they do not depend on t or f; that is, if $c_m(t,f)$ is independent of t and f for $1 \le m \le M$. Since $\tilde{E}_k$ is the <u>transmitted</u> signal energy in the k-th pulse, then

$$\tilde{E}_k\, c_m^2(t_k,f_k) \equiv D_{km} = \text{deterministic \underline{received} signal energy in m-th component of k-th pulse,} \tag{50}$$

$$\tilde{E}_k \sum_{m=1}^{M} c_m^2(t_k,f_k) = \sum_{m=1}^{M} D_{km} \equiv D_k = \text{deterministic received signal energy in k-th pulse ,} \tag{51}$$

$$\tilde{E}_k\, R_{mm}(0,0) \equiv E_{km} = \text{average random \underline{received} signal energy in m-th component of k-th pulse,} \tag{52}$$

$$\tilde{E}_k \sum_{m=1}^{M} R_{mm}(0,0) = \sum_{m=1}^{M} E_{km} \equiv E_k = \text{average random received signal energy in k-th pulse .} \tag{53}$$

The alternating (zero-mean) component of power scaling q_k is then given by (44), (45), and (49) as

$$\tilde{q}_k = q_k - \overline{q_k} =$$

$$= \sum_{m=1}^{M} \left[2\, c_m(t_k,f_k)\, g_m(t_k,f_k) + g_m^2(t_k,f_k) - R_{mm}(0,0)\right] . \tag{54}$$

The covariance between a pair of power-fading variates is

$$\tilde{R}_{kj} = \overline{\tilde{q}_k\, \tilde{q}_j} , \tag{55}$$

while the covariance coefficient between q_k and q_j is

$$\rho_{kj} = \frac{\tilde{R}_{kj}}{\left(\tilde{R}_{kk}\ \tilde{R}_{jj}\right)^{\frac{1}{2}}} \ . \tag{56}$$

Thus, the fundamental calculation required for determination of covariance coefficient ρ_{kj} is $\tilde{R}_{kj}$ as defined by (55) in conjunction with (54).

When we substitute (54) in (55) and use the fact that $\{g_m(t,f)\}$ are zero-mean joint Gaussian processes, the first-order and third-order moments involving $\{g_m(t,f)\}$ are zero, while the fourth-order moment can be broken down into a sum of products of second-order moments, leading to covariance

$$\tilde{R}_{kj} = 2\sum_{m,n=1}^{M} R_{mn}^2(t_k-t_j, f_k-f_j) + $$
$$+ 4\sum_{m,n=1}^{M} R_{mn}(t_k-t_j, f_k-f_j)\ c_m(t_k,f_k)\ c_n(t_j,f_j) \ . \tag{57}$$

In particular, the variance of power-fading variate q_k is

$$\tilde{R}_{kk} = \overline{\tilde{q}_k^2} = 2\sum_{m,n=1}^{M} R_{mn}^2(0,0) + $$
$$+ 4\sum_{m,n=1}^{M} R_{mn}(0,0)\ c_m(t_k,f_k)\ c_n(t_k,f_k) \ . \tag{58}$$

The covariance coefficient ρ_{kj} between q_k and q_j is obtained upon substitution of general results (57) and (58) into (56).

A large number of special cases can be obtained from the general formulation of fading in (45) and (47). Below, we list seven special cases that will occupy most of our attention in the remainder of this report, and which will consistently be referred to according to their number in the sequel.

SPECIAL CASE 1: Zero means $\{c_m(t,f)\}$

$$c_m(t,f) = 0 \quad \text{for } 1 \le m \le M \,. \tag{59}$$

$$\tilde{R}_{kj} = 2 \sum_{m,n=1}^{M} R_{mn}^2(t_k - t_j, f_k - f_j) \,,$$

$$\tilde{R}_{kk} = 2 \sum_{m,n=1}^{M} R_{mn}^2(0,0) \,. \tag{60}$$

SPECIAL CASE 2: Uncorrelated components $\{g_m(t,f)\}$

$$R_{mn}(\tau,\nu) = R_{mm}(\tau,\nu)\ \delta_{mn} \quad \text{for } 1 \le m,\ n \le M \,. \tag{61}$$

$$\tilde{R}_{kj} = 2 \sum_{m=1}^{M} R_{mm}^2(t_k - t_j, f_k - f_j) +$$

$$+ 4 \sum_{m=1}^{M} R_{mm}(t_k - t_j, f_k - f_j)\ c_m(t_k, f_k)\ c_m(t_j, f_j) \,,$$

$$\tilde{R}_{kk} = 2 \sum_{m=1}^{M} R_{mm}^2(0,0) + 4 \sum_{m=1}^{M} R_{mm}(0,0)\ c_m^2(t_k, f_k) \,. \tag{62}$$

It is very important to note that correlated fading still exists between the received signal pulses; that is, covariance $\tilde{R}_{kj}$ in (62) is <u>not</u> zero, despite uncorrelated property (61) in case 2.

SPECIAL CASE 3: Zero means and uncorrelated components

$$\tilde{R}_{kj} = 2 \sum_{m=1}^{M} R_{mm}^2(t_k-t_j, f_k-f_j) \ ,$$

$$\tilde{R}_{kk} = 2 \sum_{m=1}^{M} R_{mm}^2(0,0) \ ,$$

$$\rho_{kj} = \frac{\sum_{m=1}^{M} R_{mm}^2(t_k-t_j, f_k-f_j)}{\sum_{m=1}^{M} R_{mm}^2(0,0)} \ . \tag{63}$$

SPECIAL CASE 4: Uncorrelated components with identical covariances $\{R_{mn}(\tau,\nu)\}$

$$R_{mn}(\tau,\nu) = R_{11}(\tau,\nu)\ \delta_{mn} \quad \text{for all } m,\ n \ . \tag{64}$$

$$\tilde{R}_{kj} = 2M\ R_{11}^2(t_k-t_j, f_k-f_j) +$$

$$+ 4\ R_{11}(t_k-t_j, f_k-f_j) \sum_{m=1}^{M} c_m(t_k, f_k)\ c_m(t_j, f_j) \ ,$$

$$\tilde{R}_{kk} = 2M\ R_{11}^2(0,0) + 4\ R_{11}(0,0) \sum_{m=1}^{M} c_m^2(t_k, f_k) \ . \tag{65}$$

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

$$\tilde{R}_{kj} = 2M\, R_{11}^2(t_k - t_j, f_k - f_j) \ ,$$

$$\tilde{R}_{kk} = 2M\, R_{11}^2(0,0) \ ,$$

$$\rho_{kj} = \left(\frac{R_{11}(t_k - t_j, f_k - f_j)}{R_{11}(0,0)}\right)^2 \ . \tag{66}$$

In this special case, the covariance coefficient between power-fading variates $q_k = q(t_k, f_k)$ and $q_j = q(t_j, f_j)$ is the square of the covariance coefficient between the amplitude-fading variates $g_m(t_k, f_k)$ and $g_m(t_j, f_j)$; see (47). Due to the identical covariances, (64), this latter covariance coefficient is the same for every m in the range $1 \le m \le M$.

SPECIAL CASE 6: Uncorrelated components with proportional covariances

This is a generalization of special case 4; it allows the random components $\{g_m(t,f)\}$ in (45) to have different strengths, as might be encountered in a fading medium. (Note: In the following, the constant $r^{(m)}$ is the average relative power measure of the m-th random fading component; it must not be confused with the random variable r_k which is the amplitude-scaling on the k-th signal pulse in (22).)

$$R_{mn}(\tau,\nu) = R_{11}(\tau,\nu)\ r^{(m)}\ \delta_{mn}\ , \qquad r^{(1)} = 1\ . \tag{67}$$

$$\tilde{R}_{kj} = 2\ R_{11}^2(t_k-t_j,f_k-f_j) \sum_{m=1}^{M} r^{(m)^2} +$$

$$+\ 4\ R_{11}(t_k-t_j,f_k-f_j) \sum_{m=1}^{M} r^{(m)}\ c_m(t_k,f_k)\ c_m(t_j,f_j)\ ,$$

$$\tilde{R}_{kk} = 2\ R_{11}^2(0,0) \sum_{m=1}^{M} r^{(m)^2} + 4\ R_{11}(0,0) \sum_{m=1}^{M} r^{(m)}\ c_m^2(t_k,f_k)\ . \tag{68}$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Setting the means $\{c_m(t,f)\}$ in (68) to zero,

$$\tilde{R}_{kj} = 2\ R_{11}^2(t_k-t_j,f_k-f_j) \sum_{m=1}^{M} r^{(m)^2},$$

$$\tilde{R}_{kk} = 2\ R_{11}^2(0,0) \sum_{m=1}^{M} r^{(m)^2},$$

$$\rho_{kj} = \left(\frac{R_{11}(t_k-t_j,f_k-f_j)}{R_{11}(0,0)}\right)^2\ . \tag{69}$$

Notice, in this special case, that $\{\rho_{kj}\}$ are independent of the values of $\{r^{(m)}\}$, the relative power measures of the M random fading components.

CHARACTERISTIC FUNCTION OF POWER-FADING VARIATE q(t,f)

The instantaneous power-fading process is, from (45),

$$q(t,f) = \sum_{m=1}^{M} \left[c_m(t,f) + g_m(t,f)\right]^2 = X^T X , \tag{70}$$

where M×1 Gaussian random column vector

$$X = \left[c_1(t,f) + g_1(t,f) \quad . \; . \; . \quad c_M(t,f) + g_M(t,f)\right]^T . \tag{71}$$

We now appeal directly to the very general result derived in appendix B, for the characteristic function of a quadratic form and linear form, and identify the quantities there according to

$$N = M , \quad B = I , \quad A = 0 , \quad E = \left[c_1(t,f) \quad . \; . \; . \quad c_M(t,f)\right]^T ,$$

$$C = \mathrm{Cov}(X) = \left[\overline{g_m(t,f)\; g_n(t,f)}\right]_1^M = \left[R_{mn}(0,0)\right]_1^M . \tag{72}$$

It is important to note that M×M symmetric covariance matrix C is <u>not</u> a function of t,f, under the stationarity assumption (47) for all M random processes $\{g_m(t,f)\}$.

According to (B-10), we must solve the standard characteristic-value matrix equation

$$C\, Q = Q\, \Lambda , \tag{73}$$

for M×M eigenvalue matrix Λ and normalized modal matrix Q, where

$$\Lambda = \mathrm{diag}\left[\lambda_1 \; \ldots \; \lambda_M\right] , \quad Q = \left[V_1 \; \ldots \; V_M\right] , \quad V_m = \left[v_{m1} \; \ldots \; v_{mM}\right]^T , \tag{74}$$

and M×1 column vector V_m is the m-th eigenvector with components

$\{v_{mn}\}$, $1 \le n \le M$. Then (B-18) with (74) and (72) yields deterministic parameters

$$\varepsilon_m = V_m^T \, E = \sum_{n=1}^{M} v_{mn} \, c_n(t,f) \equiv \varepsilon_m(t,f) \; , \quad \alpha_m = 0 \; , \tag{75}$$

for $1 \le m \le M$. We can now use (B-20) to obtain the characteristic function of power-fading variate $q(t,f)$ as

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi\lambda_m\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{\varepsilon_m^2(t,f)}{1 - i2\xi\lambda_m}\right] . \tag{76}$$

The eigenvectors $\{V_m\}$ and nonzero means $\{c_m(t,f)\}$ enter this result through the terms $\{\varepsilon_m(t,f)\}$ defined in (75). This general result will be simplified, below, to the seven special cases that were presented earlier in (59) - (69).

By expanding the logarithm of general result (76) in a power series in $i\xi$, the cumulants of $q(t,f)$ are found to be

$$\chi_q(p) = 2^{p-1} \, (p-1)! \sum_{m=1}^{M} \left(\lambda_m + p \, \varepsilon_m^2(t,f)\right) \lambda_m^{p-1} \quad \text{for } p \ge 1 \; . \tag{77}$$

For $p = 1$ and $p = 2$, these may be verified to equal the results in (49) and (58); the pertinent matrix manipulations are indicated in appendix B for the more general case where linear form $A \neq 0$. The quantities $\{\lambda_m\}$ are the eigenvalues of covariance matrix C in (72), while parameters $\{\varepsilon_m(t,f)\}$ are given by (75) in terms of eigenvectors $\{V_m\}$ of matrix C and mean vector $E = E(t,f)$ in (72).

SPECIAL CASE 1: Zero means; see (59)

Since $c_m(t,f) = 0$, then $\varepsilon_m(t,f) = 0$ from (75), and the characteristic function in (76) reduces to

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi\lambda_m\right)\right]^{-\frac{1}{2}} . \tag{78}$$

SPECIAL CASE 2: Uncorrelated components; see (61)

Here, from (72), we have

$$C = \text{diag}[R_{11}(0,0) \ . \ . \ . \ R_{MM}(0,0] \ . \tag{79}$$

Then, the eigenvalue matrix Λ in (73) must be identical to C, in which case (73) becomes C Q = Q C, where C is diagonal. This forces Q to be diagonal also; finally, normalization of Q yields Q = I. There follows, from (74) - (76),

$$v_{mn} = \delta_{mn} \ , \qquad \varepsilon_m = c_m(t,f) \ , \tag{80}$$

and characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1-i2\xi R_{mm}(0,0)\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{c_m^2(t,f)}{1-i2\xi R_{mm}(0,0)}\right] . \tag{81}$$

SPECIAL CASE 3: Zero means and uncorrelated components

Here, we simply set the constants in (81) equal to zero:

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi \ R_{mm}(0,0)\right)\right]^{-\frac{1}{2}} . \tag{82}$$

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Now we use (64) on result (81) to get

$$f_q(\xi) = \left(1-i2\xi R_{11}(0,0)\right)^{-M/2} \exp\left[\frac{i\xi}{1-i2\xi R_{11}(0,0)} \sum_{m=1}^{M} c_m^2(t,f)\right]. \quad (83)$$

It is important to observe in this case that only the sum of the squares of the means, $\{c_m(t,f)\}$, matters in so far as the characteristic function of power scaling $q(t,f)$ is concerned.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means in (83) equal to zero, there follows

$$f_q(\xi) = \left(1 - i2\xi\, R_{11}(0,0)\right)^{-M/2}. \quad (84)$$

This special case is equivalent to [12; (A-21)] when we take the parameter m there equal to M/2.

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

From (67), there follows

$$R_{mn}(0,0) = R_{11}(0,0)\, r^{(m)}\, \delta_{mn}\ , \qquad r^{(1)} = 1\ . \tag{85}$$

Use of this relation on (81) yields characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1-i2\xi R_{11}(0,0)r^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{c_m^2(t,f)}{1-i2\xi R_{11}(0,0)r^{(m)}}\right]. \tag{86}$$

The cumulants of power scaling q(t,f) can be obtained by expanding the logarithm of characteristic function (86) in a power series in $i\xi$; there follows

$$\frac{\chi_q(p)}{(p-1)!\ 2^{p-1}} = R_{11}^{p}(0,0) \sum_{m=1}^{M} r^{(m)p} + p\, R_{11}^{p-1}(0,0) \sum_{m=1}^{M} r^{(m)p-1}\ c_m^2(t,f)$$

$$\text{for } p \geq 1\ . \tag{87}$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Setting the means in (86) to zero, we have

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi\ R_{11}(0,0)\ r^{(m)}\right)\right]^{-\frac{1}{2}}. \tag{88}$$

Actual numerical examples displaying typical probability density functions of power-scaling random variable q, for a variety of different parameter values, will be presented in a later section.

CHARACTERISTIC FUNCTION OF POWER-SUM VARIATE S

The power-sum variate S is given in (37) and (43) as the sum of the power scalings $q_k = q(t_k, f_k)$ over the total of K pulses transmitted and received:

$$S = \sum_{k=1}^{K} q(t_k, f_k) = \sum_{k=1}^{K} \sum_{m=1}^{M} \left[c_m(t_k, f_k) + g_m(t_k, f_k)\right]^2 , \qquad (89)$$

where we used (44) and (45). This double sum can be recognized as a quadratic form in KM correlated nonzero-mean Gaussian random variables. Therefore, the general approach in appendix B can be used directly to find the characteristic function of S for <u>any</u> statistical dependencies between the M zero-mean Gaussian processes $\{g_m(t,f)\}$ and <u>any</u> layout of the K points $\{t_k, f_k\}$ in the time,frequency plane. In order to realize form (89), we identify variate q = S, constant N = KM, and matrices B = I, A = 0 in appendix B. Mean column vector E in (B-1) is KM×1, while symmetric covariance matrix C is KM×KM. Then (B-10) becomes the standard characteristic-value matrix equation [18; section 1.13]

$$C\,Q = Q\,\Lambda , \quad C = \left[\overline{g_m(t_k, f_k)\; g_n(t_j, f_j)}\right] = \left[R_{mn}(t_k - t_j, f_k - f_j)\right], \qquad (90)$$

where KM×KM modal matrix $Q = [V_1 \; . \; . \; . \; V_{KM}]$ from (B-11). Also, there follows from (B-18),

$$\underline{E} = Q^T E , \quad \varepsilon_n = V_n^T E \quad \text{for } 1 \le n \le KM ; \quad \alpha_n = 0 . \qquad (91)$$

The characteristic function of sum S in (89) then follows from (B-20) as

$$f_S(\xi) = \left[\prod_{n=1}^{KM}\left(1 - i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^{KM} \frac{\varepsilon_n^2}{1 - i2\xi\lambda_n}\right] . \tag{92}$$

This general result will now be specialized to the seven cases of particular interest here.

SPECIAL CASE 1: Zero means; see (59)

When $c_m(t,f) = 0$ for $1 \le m \le M$, then mean vector E is zero, meaning that its components $\{\varepsilon_n\}$ in (91) are zero. The result in (92) then reduces to

$$f_S(\xi) = \left[\prod_{n=1}^{KM}\left(1 - i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} . \tag{93}$$

Only the KM eigenvalues of KM×KM covariance matrix C of the KM random variables $\{g_m(t_k,f_k)\}$ need to be evaluated in this case.

SPECIAL CASE 2: Uncorrelated components; see (61)

From this point on, we shall be interested in the more restricted case where component Gaussian process $g_m(t,f)$ is uncorrelated with (independent of) process $g_n(t',f')$ for $m \ne n$, regardless of the values of t,f and t',f'; this case was also considered earlier in (61). This will probably encompass most situations of practical interest; furthermore, it still allows for correlated fading between the K received signal pulses. That is, covariance $\tilde{R}_{jk}$ in (62), between power scalings q_k and q_j,

need not be zero, despite uncorrelated property (61) between components $\{g_m(t,f)\}$ in this special case 2.

In this case, a simpler and more direct approach is possible and is adopted. We begin by expressing (89) as

$$S = \sum_{m=1}^{M} S^{(m)} \ , \tag{94}$$

where we have defined the M random variables

$$S^{(m)} = \sum_{k=1}^{K} \left[c_m(t_k,f_k) + g_m(t_k,f_k)\right]^2 \quad \text{for } 1 \le m \le M \ . \tag{95}$$

This superscript m notation is adopted in order to readily distinguish the fading component numbers, $1 \le m \le M$, from the time,frequency signal pulse numbe~s, $1 \le k \le K$; see (45) versus figure 1.

Now, it is important to observe, for this special case 2, that these latter M random variables $\{S^{(m)}\}$ are statistically independent of each other, allowing us to develop the characteristic function of sum S in (94) in the finite product form

$$f_S(\xi) = \prod_{m=1}^{M} f^{(m)}(\xi) \ , \tag{96}$$

where the m-th characteristic function is given by average

$$f^{(m)}(\xi) = \overline{\exp\left(i\xi S^{(m)}\right)} \ , \tag{97}$$

in terms of quadratic sum $S^{(m)}$ in (95).

In order to ascertain the characteristic function of sum $S^{(m)}$, we define K×1 Gaussian column vector $Y^{(m)}$ according to

$$Y^{(m)} = \left[c_m(t_1,f_1) + g_m(t_1,f_1) \; . \; . \; . \; c_m(t_K,f_K) + g_m(t_K,f_K)\right]^T . \quad (98)$$

Then, $S^{(m)}$ in (95) can be written in the quadratic form

$$S^{(m)} = Y^{(m)^T} Y^{(m)} . \quad (99)$$

We now appeal to the general results in appendix B and identify the quantities there according to $q = S^{(m)}$, $N = K$, and

$$B = I \;, \quad A = 0 \;, \quad E = E^{(m)} = \overline{Y^{(m)}} = \left[c_m(t_1,f_1) \; . \; . \; . \; c_m(t_K,f_K)\right]^T ,$$

$$C = C^{(m)} = \mathrm{Cov}\left(Y^{(m)}\right) = \left[\overline{g_m(t_k,f_k)\; g_m(t_j,f_j)}\right]_1^K = \left[R_{mm}(t_k-t_j,f_k-f_j)\right]_1^K \quad (100)$$

The K diagonal elements of matrix $C^{(m)}$ are all equal to $R_{mm}(0,0)$.

According to (B-10), we must now solve, for each value of m in the range $1 \le m \le M$, the K×K characteristic-value equation

$$C^{(m)} Q^{(m)} = Q^{(m)} \Lambda^{(m)} , \quad (101)$$

for K×K eigenvalue matrix $\Lambda^{(m)}$ and corresponding K×K normalized modal matrix $Q^{(m)}$, where

$$\Lambda^{(m)} = \mathrm{diag}\left[\lambda_1^{(m)} \; . \; . \; . \; \lambda_K^{(m)}\right] , \quad Q^{(m)} = \left[V_1^{(m)} \; . \; . \; . \; V_K^{(m)}\right] ,$$

$$V_k^{(m)} = \left[v_{k1}^{(m)} \; . \; . \; . \; v_{kK}^{(m)}\right]^T , \quad (102)$$

and K×1 vector $V_k^{(m)}$ is the k-th eigenvector with components $\{v_{kj}^{(m)}\}$, $1 \le j \le K$. Then, (B-18) with (102) and (100) yields

$$\varepsilon_k^{(m)} = V_k^{(m)^T} E^{(m)} = \sum_{j=1}^{K} v_{kj}^{(m)} c_m(t_j,f_j) , \quad \alpha_k^{(m)} = 0 . \quad (103)$$

We can now use (B-20) to obtain the characteristic function of variate $S^{(m)}$ in (99) as

$$f^{(m)}(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\lambda_k^{(m)}}\right] . \quad (104)$$

The desired characteristic function of power-sum variate S in (89) is then given by (96), applied to (104):

$$f_S(\xi) = \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\lambda_k^{(m)}}\right] . \quad (105)$$

This characteristic function of sum S is a very general result, applicable to the case of uncorrelated components $\{g_m(t,f)\}$; however, it does require the solution of M matrix equations of the form of (101), each matrix being of size K×K. Nevertheless, this approach is significantly simpler than solving the one large KM×KM matrix equation (90).

SPECIAL CASE 3: Zero means and uncorrelated components

When the means $\{c_m(t,f)\}$ are zero, (103) yields $\varepsilon_k^{(m)} = 0$ and the characteristic function in (105) reduces to just the product

$$f_S(\xi) = \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k^{(m)}\right)\right]^{-\frac{1}{2}} . \quad (106)$$

Eigenvalues $\lambda_1^{(m)}, \ldots, \lambda_K^{(m)}$ are found from K×K covariance matrix $C^{(m)}$ in (100); this solution must be repeated for $1 \le m \le M$.

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Here, covariance matrix $C^{(m)}$ in (100) is independent of m; in particular, we now have the single K×K covariance matrix

$$C = \left[R_{11}(t_k - t_j, f_k - f_j)\right]_1^K . \tag{107}$$

This leads to solution matrices $Q^{(m)}$ and $\Lambda^{(m)}$ in (101) which are also independent of m; that is, (101) becomes the single K×K characteristic-value matrix equation

$$C\ Q = Q\ \Lambda . \tag{108}$$

Thus, the eigenvalues and eigenvectors in (102) are independent of m. However, the constants $\{\varepsilon_k^{(m)}\}$ in (103) still depend on m through their dependencies on means $\{c_m(t,f)\}$; that is, from (103),

$$\varepsilon_k^{(m)} = V_k^T\ E^{(m)} = \sum_{j=1}^{K} v_{kj}\ c_m(t_j, f_j) . \tag{109}$$

The collection of all these conclusions enables us to reduce the characteristic function (105) of S to the compact closed form

$$f_S(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k\right)\right]^{-M/2} \exp\left[i\xi \sum_{k=1}^{K} \frac{h_k}{1 - i2\xi\lambda_k}\right] , \tag{110}$$

where we defined the K constants

$$h_k = \sum_{m=1}^{M} \varepsilon_k^{(m)^2} \quad \text{for } 1 \le k \le K . \tag{111}$$

If the number, M, of components in fading model (45) happens to be even, the calculation of (110) does not involve a square root, thereby further simplifying its numerical evaluation. The collapsing of the KM constants $\{\varepsilon_k^{(m)}\}$ into a smaller set of K constants, by means of the sums of squares in (111), is the analog of the result in (83) for an individual fading variate $q(t,f)$ in special case 4.

As a special subcase here, suppose that the M deterministic components $\{c_m(t,f)\}$ in (45) are independent of t and f; that is,

$$c_m(t_k,f_k) = c_m \quad \text{for } 1 \le k \le K . \tag{112}$$

This means that the constants are independent of the locations $\{t_k,f_k\}$ of the signal pulses in the t,f plane, although they can still depend on the component number m. Then, there follows from (109) and (111),

$$\varepsilon_k^{(m)} = c_m \sum_{j=1}^{K} v_{kj} , \qquad h_k = \left(\sum_{m=1}^{M} c_m^2\right) \left(\sum_{j=1}^{K} v_{kj}\right)^2 . \tag{113}$$

That is, the characteristic function of sum S in (110) depends on the constants $\{c_m\}$ only through their sum of squares. This latter property holds true only when special subcase (112) is valid.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means $\{c_m(t,f)\}$ to zero, it is seen from (109) and (111) that characteristic function (110) reduces to

$$f_S(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k\right)\right]^{-M/2} . \tag{114}$$

Now, only the K eigenvalues $\{\lambda_k\}$ of K×K covariance matrix C in (107) need to be evaluated. This result in (114) is essentially identical with [10; (D-14)]. For the special case of M = 2, this same result for the characteristic function of S can be shown to follow from [11; (20)].

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

From (67) and (100), we find that K×K covariance matrix

$$C^{(m)} = r^{(m)}\, C \ , \quad \text{where} \quad C = \left[R_{11}(t_k - t_j, f_k - f_j)\right]_1^K , \tag{115}$$

and $r^{(1)} = 1$. Then (101) yields

$$Q^{(m)} = Q \ , \quad \Lambda^{(m)} = r^{(m)}\, \Lambda \ , \tag{116}$$

where Q and Λ are the solutions to the single K×K characteristic-value matrix equation

$$C\, Q = Q\, \Lambda \ . \tag{117}$$

We now refer to (103) to get constants

$$\varepsilon_k^{(m)} = V_k^T \, E^{(m)} = \sum_{j=1}^{K} v_{kj} \, c_m(t_j, f_j) \ . \tag{118}$$

Then (104) yields the characteristic function of $S^{(m)}$ as

$$f^{(m)}(\xi) = \left[\prod_{k=1}^{K} \left(1 - i2\xi\lambda_k r^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\lambda_k r^{(m)}}\right] . \tag{119}$$

Finally, (96) gives the characteristic function of sum S as

$$f_S(\xi) = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi \ \lambda_k \ r^{(m)}\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi \ \lambda_k \ r^{(m)}}\right] . \tag{120}$$

Here, even if the M deterministic components $\{c_m(t,f)\}$ were independent of t and f, as previously assumed in (112), the characteristic function of S would <u>not</u> depend simply on the sum of squares $\sum_m c_m^2$. To see this, substitute the left member of (113) in (120).

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Set the means $\{c_m(t,f)\}$ to zero in (118), at which point the characteristic function for S in (120) simplifies to

$$f_S(\xi) = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\ \lambda_k\ r^{(m)}\right)\right]^{-\frac{1}{2}} . \qquad (121)$$

Eigenvalues $\{\lambda_k\}$ are the diagonal elements of eigenvalue matrix Λ in K×K matrix equation (117), where covariance matrix C is given by (115).

CUMULANTS OF SUM S

For some purposes, such as approximating the probability density function or exceedance distribution function of S, the cumulants are useful; see, for example, [17]. By expanding the natural logarithm of the general characteristic function in (92) in a power series in $i\xi$, the cumulants of S are readily found:

$$\chi_S(p) = (p-1)!\ 2^{p-1} \sum_{n=1}^{KM} \lambda_n^{p-1}\left(\lambda_n + p\varepsilon_n^2\right) \quad \text{for } p \ge 1 . \qquad (122)$$

The cumulants for the seven special cases considered above could also be derived. However, we will only present the results for special case 6, namely uncorrelated components with proportional covariances; see (67). Again, expanding the logarithm of characteristic function (120) in a series in $i\xi$, there follows a slight simplification of (122) for the cumulants of S:

$$\chi_S(p) = (p-1)!\ 2^{p-1} \sum_{m=1}^{M} r^{(m)p} \sum_{k=1}^{K} \lambda_k^p +$$

$$+\ p!\ 2^{p-1} \sum_{m=1}^{M} \sum_{k=1}^{K} \left(\lambda_k\ r^{(m)}\right)^{p-1} \varepsilon_k^{(m)2} \quad \text{for } p \ge 1 . \qquad (123)$$

CHARACTERISTIC FUNCTION OF PROCESSOR OUTPUT γ

With the characteristic function of sum S in hand, we can now return to the desired result (42) for the characteristic function $f_\gamma(\xi)$ of processor output γ, when conditions (38) are satisfied. Substitution of general result (92) into (42) yields

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{n=1}^{KM}\left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \frac{2\tilde{E}}{N_o}\sum_{n=1}^{KM} \frac{\varepsilon_n^2}{1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)}\right] . \qquad (124)$$

By expanding the logarithm of (124) in a power series in $i\xi$, the cumulants $\chi_\gamma(p)$ of processor output γ are found to be given by

$$\frac{\chi_\gamma(p)}{(p-1)!\ 2^{p-1}} = \sum_{n=1}^{KM}\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)^p + \frac{2\tilde{E}}{N_o}\, p \sum_{n=1}^{KM}\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)^{p-1}\varepsilon_n^2 - K(M-2)$$

$$\text{for } p \ge 1 . \qquad (125)$$

SPECIAL CASE 1: Zero means; see (59)

Upon setting means $\{c_m(t,f)\}$ to zero in (124), the constants $\{\varepsilon_n\}$ defined in (91) become zero and (124) reduces to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{n=1}^{KM}\left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)\right)\right]^{-\frac{1}{2}} . \qquad (126)$$

Only the eigenvalues $\{\lambda_n\}$ of KM×KM covariance matrix C of the KM random variables $\{g_m(t_k,f_k)\}$ need to be evaluated in this case.

SPECIAL CASE 2: Uncorrelated components; see (61)

The comments in the bottom paragraph of page 32 and the sequel are relevant at this point and should be reviewed. For convenience, we will utilize a normalized version of covariance matrix $C^{(m)}$ defined in (100):

$$\underline{C}^{(m)} = \frac{C^{(m)}}{R_{mm}(0,0)} = \left[\frac{R_{mm}(t_k-t_j,f_k-f_j)}{R_{mm}(0,0)}\right]_1^K . \tag{127}$$

The corresponding eigenvalue matrix $\underline{\Lambda}^{(m)}$ of normalized K×K covariance matrix $\underline{C}^{(m)}$ is then given by

$$\underline{\Lambda}^{(m)} = \frac{\Lambda^{(m)}}{R_{mm}(0,0)} \equiv \mathrm{diag}\left[\underline{\lambda}_1^{(m)} \cdots \underline{\lambda}_K^{(m)}\right] \tag{128}$$

in terms of eigenvalue matrix $\Lambda^{(m)}$ in (101), which now becomes

$$\underline{C}^{(m)}\, Q^{(m)} = Q^{(m)}\, \underline{\Lambda}^{(m)} . \tag{129}$$

Normalized modal matrix $Q^{(m)}$ is unchanged from (101). The relationship between the eigenvalues in (102) and (128) is

$$\lambda_k^{(m)} = \underline{\lambda}_k^{(m)}\, R_{mm}(0,0) . \tag{130}$$

When we employ special case 2 result (105) in (42), the characteristic function of processor output γ is given by

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_k^{(m)}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi\,\frac{2\tilde{E}}{N_o}\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{{\varepsilon_k^{(m)}}^2}{1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_k^{(m)}\right)}\right] . \tag{131}$$

However, a combination of (130) and (52) yields

$$\frac{2\tilde{E}}{N_o} \lambda_k^{(m)} = \frac{2\tilde{E}}{N_o} R_{mm}(0,0)\, \underline{\lambda}_k^{(m)} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)} , \tag{132}$$

where we have taken note that the same signal energy $\tilde{E}$ was transmitted on all K pulses; see (38). That is, E_{1m} is the average random <u>received</u> signal energy in the m-th component of any one of the pulses. The quantity $2E_{1m}/N_o$ is a measure of the average random received "signal-to-noise ratio" in the m-th component of one pulse. At the same time, from (103),

$$\frac{2\tilde{E}}{N_o} \varepsilon_k^{(m)^2} = \left[\sum_{j=1}^{K} v_{kj}^{(m)}\, c_m(t_j,f_j) \left(\frac{2\tilde{E}}{N_o}\right)^{\frac{1}{2}}\right]^2 . \tag{133}$$

But since $c_m(t,f) \geq 0$ without loss of generality (see (46)), we have

$$c_m(t_j,f_j) \left(\frac{2\tilde{E}}{N_o}\right)^{\frac{1}{2}} = \left(\frac{2\tilde{E}}{N_o}\, c_m^2(t_j,f_j)\right)^{\frac{1}{2}} = \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} , \tag{134}$$

where D_{jm} is the <u>deterministic</u> <u>received</u> signal energy in the m-th component of the j-th pulse, as defined in (50). Then (133) can be expressed as

$$\frac{2\tilde{E}}{N_o} \varepsilon_k^{(m)^2} = \left[\sum_{j=1}^{K} v_{kj}^{(m)} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}}\right]^2 \equiv \underline{\varepsilon}_k^{(m)^2} . \tag{135}$$

The quantity $2D_{jm}/N_o$ is a measure of the received deterministic signal-to-noise ratio in the m-th component of the j-th pulse.

Upon combining these definitions, (131) is modified to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\underline{\varepsilon}_k^{(m)^2}}{1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)}\right] . \tag{136}$$

Here, eigenvalues $\underline{\lambda}_1^{(m)},\ldots,\underline{\lambda}_K^{(m)}$ are those of the K×K normalized covariance matrix $\underline{C}^{(m)}$ defined in (127). The constants $\underline{\varepsilon}_k^{(m)}$ are given, according to (135), by

$$\underline{\varepsilon}_k^{(m)} = \sum_{j=1}^{K} v_{kj}^{(m)} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} . \tag{137}$$

The cumulants of processor output γ in this case are given by

$$\frac{\chi_\gamma(p)}{(p-1)!\ 2^{p-1}} = \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)^p +$$

$$+ p \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)^{p-1} \underline{\varepsilon}_k^{(m)^2} - K(M-2) \quad \text{for } p \geq 1 . \tag{138}$$

SPECIAL CASE 3: Zero means and uncorrelated components

When means $\{c_m(t,f)\}$ are zero, then (50) and (137) yield $D_{km} = 0$ and $\underline{\varepsilon}_k^{(m)} = 0$, thereby causing (136) to reduce to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)\right)\right]^{-\frac{1}{2}} . \tag{139}$$

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Now, normalized covariance matrix $\underline{C}^{(m)}$ in (127) is independent of m; in particular, we have the single K×K covariance matrix

$$\underline{C} = \frac{C}{R_{11}(0,0)} = \left[\frac{R_{11}(t_k-t_j,f_k-f_j)}{R_{11}(0,0)}\right]_1^K . \tag{140}$$

This leads to solution matrices $Q^{(m)}$ and $\underline{\Lambda}^{(m)}$ in (129) which are also independent of m; that is, (129) becomes the single K×K characteristic-value matrix equation

$$\underline{C}\ Q = Q\ \underline{\Lambda}\ ; \qquad \underline{\Lambda} = \Lambda/R_{11}(0,0) = \mathrm{diag}[\underline{\lambda}_1,\ldots,\underline{\lambda}_K] . \tag{141}$$

Thus, the eigenvalues in (128) and the eigenvectors in (102) are independent of m. However, the constants $\underline{\varepsilon}_k^{(m)}$ in (137) still depend on m through their dependence on deterministic received signal energies $\{D_{jm}\}$; that is, from (137), now

$$\underline{\varepsilon}_k^{(m)} = \sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} . \tag{142}$$

Also, from (52), (38), and (64), we now have

$$E_{1m} = \tilde{E}_1\ R_{mm}(0,0) = \tilde{E}\ R_{11}(0,0) = E_{11} , \tag{143}$$

which is the average random received signal energy in one component of one pulse. (In this special case 4, we have the alternative result $E_{11} = E_1/M$ from (53).)

The collection of these conclusions enables us to express the characteristic function (136) of processor output γ in the closed compact form

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}M} \times$$

$$\times \exp\left[i\xi \sum_{k=1}^{K} \frac{\underline{h}_k}{1 - i2\xi\left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)}\right] , \qquad (144)$$

where we defined the K constants

$$\underline{h}_k = \sum_{m=1}^{M} \underline{\varepsilon}_k^{(m)^2} = \sum_{m=1}^{M} \left\{\sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}}\right\}^2 \quad \text{for } 1 \le k \le K . \qquad (145)$$

Use of (142) was made here. The cumulants of output γ are now

$$\frac{\chi_\gamma(p)}{(p-1)!\; 2^{p-1}} = M \sum_{k=1}^{K}\left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)^p +$$

$$+ p \sum_{k=1}^{K} \underline{h}_k \left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)^{p-1} - K(M-2) \quad \text{for } p \ge 1 . \qquad (146)$$

As a special subcase, suppose that the M deterministic components $\{c_m(t,f)\}$ in fading model (45) are independent of t and f; see (112). Then, from (50) and (38),

$$D_{km} = \tilde{E}_k\, c_m^2(t_k,f_k) = \tilde{E}\, c_m^2 = D_{1m} , \qquad (147)$$

where D_{1m} is the received deterministic signal energy in the m-th component of any one of the pulses. This result allows the constants in (145) to be simplified to

$$\underline{h}_k = \left(\sum_{m=1}^{M} \frac{2D_{1m}}{N_o}\right) \left(\sum_{j=1}^{K} v_{kj}\right)^2 . \tag{148}$$

Thus, in this special subcase, the constants $\{\underline{h}_k\}$ depend only on the sum of the received deterministic signal energies on all the components.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means $\{c_m(t,f)\}$ to zero, it follows from (50), (145), and (144) that the characteristic function of processor output γ is

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{11}}{N_o} \underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}M} . \tag{149}$$

Eigenvalues $\underline{\lambda}_1, \ldots, \underline{\lambda}_K$ correspond to K×K normalized covariance matrix $\underline{C}$ given by (140), while E_{11} is the average random received signal energy in one component of one pulse. This exact result replaces the approximation in [12; (8)].

In the special case of M = 2, that is, two components in fading model (45), the result in (149) reduces to

$$f_\gamma(\xi) = \left[\prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{11}}{N_o} \underline{\lambda}_k\right)\right)\right]^{-1} . \tag{150}$$

This expression agrees with [11; (24)], except for a scaling of output γ by a factor of 2. There was no need here to evaluate multiple integrals as encountered in [11; (21) - (22)].

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

If we combine the general relation in (42), for the characteristic function of processor output γ, with the characteristic function in (120) for the sum S in this special case 6, we obtain the form

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o} \lambda_k \, r^{(m)}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \, \frac{2\tilde{E}}{N_o} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o} \lambda_k \, r^{(m)}\right)}\right] . \qquad (151)$$

Now, from (67) and (100), we find that K×K covariance matrix

$$C^{(m)} = r^{(m)} \, C \; , \quad \text{where} \quad C = \left[R_{11}(t_k - t_j, f_k - f_j)\right]_1^K \; , \qquad (152)$$

and $r^{(1)} = 1$. Then (101) yields

$$Q^{(m)} = Q \; , \quad \Lambda^{(m)} = r^{(m)} \, \Lambda \; , \qquad (153)$$

where Q and Λ are the solutions to the single K×K matrix equation

$$C \, Q = Q \, \Lambda \; . \qquad (154)$$

Now, define normalized covariance matrix $\underline{C}$ as in (140), with corresponding matrix equation and eigenvalue matrix $\underline{\Lambda}$ as in (141). Then the eigenvalues are related according to

$$\lambda_k = \underline{\lambda}_k \, R_{11}(0,0) \; , \qquad (155)$$

leading to result

$$\frac{2\tilde{E}}{N_o} \lambda_k r^{(m)} = \frac{2\tilde{E}}{N_o} R_{11}(0,0) \underline{\lambda}_k r^{(m)} = \frac{2E_{11}}{N_o} \underline{\lambda}_k r^{(m)} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k , \quad (156)$$

upon use of (38), (52), and (67). The quantity E_{1m} is the average random received signal energy in the m-th component of any one pulse. Also, from (135) and (137),

$$\frac{2\tilde{E}}{N_o} \varepsilon_k^{(m)^2} = \underline{\varepsilon}_k^{(m)^2} , \qquad \underline{\varepsilon}_k^{(m)} = \sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} , \quad (157)$$

since normalized modal matrix Q is independent of component number m; see (153). Use of (156) and (157) in (151) yields the characteristic function of processor output γ in the desired form

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\underline{\varepsilon}_k^{(m)^2}}{1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)}\right] . \quad (158)$$

This result is a slight simplification of (136) which allowed uncorrelated components of arbitrary covariances; thus, eigenvalues $\{\underline{\lambda}_k\}$ and eigenvectors $\{V_k\}$ are independent of m here. Furthermore, (158) simplifies to the result in (144) when the covariances of the random fading components $\{g_m(t,f)\}$ in (45) are identical; see (64).

The cumulants $\{\chi_\gamma(p)\}$ of processor output γ are found by expanding the logarithm of (158) in a power series in $i\xi$:

$$\frac{X_\gamma(p)}{(p-1)!\;2^{p-1}} = \sum_{m=1}^{M}\sum_{k=1}^{K}\left(1 + \frac{2E_{1m}}{N_o}\,\underline{\lambda}_k\right)^{p} +$$

$$+\; p\sum_{m=1}^{M}\sum_{k=1}^{K}\left(1 + \frac{2E_{1m}}{N_o}\,\underline{\lambda}_k\right)^{p-1} \underline{\varepsilon}_k^{(m)^2} - K(M-2) \quad \text{for } p \geq 1 \;. \qquad (159)$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

When we set means $\{c_m(t,f)\}$ to zero, the characteristic function in (158) simplifies to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o}\,\underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}}. \qquad (160)$$

For purposes of review, the quantity E_{km} is the average random received signal energy in the m-th component of the k-th pulse, while D_{km} is the received deterministic signal energy in the m-th component of the k-th pulse; see (50) and (52) as well as fading model (45) - (47). Eigenvalues $\{\underline{\lambda}_k\}$ correspond to normalized covariance matrix $\underline{C}$ in (140), while coefficients $\{v_{kj}\}$ used in (157) are the corresponding eigenvectors' components; see the K×K matrix equation in (141), which yields eigenvalue matrix $\underline{\Lambda}$ and normalized modal matrix Q.

In the sequel to (42), it was noted that the more general conditional characteristic function of γ in (27) would be treated once the fading model had been described. This latter situation with general receiver weights $\{A_k\}$ and transmitted signal energies $\{\tilde{E}_k\}$ is considered in appendix C. In particular, the unconditional characteristic function of processor output γ is derived for general correlated fading components $\{g_m(t,f)\}$ and then specialized to the case of uncorrelated fading components.

EXAMPLES OF PROBABILITY DENSITY OF POWER-SCALING q(t,f)

In an earlier section, the characteristic function of power-scaling variate $q = q(t,f)$ was derived for fading model (45), yielding general result (76). This result was then simplified for the seven special cases noted there. In this section, we will present some numerical examples of typical probability density functions of q and thereby illustrate the generality and variety of fading model (45).

The only case we will consider in this section is where the M random components $\{g_m(t,f)\}$ in (45) are uncorrelated with each other and have proportional covariances. This is special case 6; see (67). Also, for notational convenience, we consider the normalized (power) random variable

$$\phi = \frac{q}{R_{11}(0,0)} = \frac{q}{\sigma_1^2} , \tag{161}$$

where

$$\sigma_m^2 \equiv \overline{g_m^2(t,f)} = R_{mm}(0,0) = R_{11}(0,0)\ r^{(m)} \quad \text{for } 1 \le m \le M . \tag{162}$$

Here, we used (47) and (67). Then, from (86), the characteristic function of random variable ϕ is

$$f_\phi(\xi) = f_q\left(\frac{\xi}{R_{11}(0,0)}\right) = \left[\prod_{m=1}^{M}\left(1-i2\xi r^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{h_m^2\ r^{(m)}}{1-i2\xi r^{(m)}}\right] , \tag{163}$$

where we defined (dimensionless) normalization constants

$$\eta_m = \frac{c_m}{\sigma_m} = \frac{c_m(t,f)}{R_{mm}(0,0)^{\frac{1}{2}}} \quad \text{for } 1 \le m \le M \,. \tag{164}$$

For M = 1, the general result in (163) simplifies to

$$f_\phi(\xi) = (1 - i2\xi)^{-\frac{1}{2}} \exp\left[\frac{i\xi\, \eta_1^2}{1 - i2\xi}\right] \,, \tag{165}$$

for which the corresponding PDF (probability density function) is

$$p_\phi(u) = (2\pi u)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(u + \eta_1^2\right)\right] \cosh(\eta_1 \sqrt{u}) \quad \text{for } u > 0 \,. \tag{166}$$

The amplitude-scaling variate, $r_k = q_k^{\frac{1}{2}}$ from (43), has the corresponding PDF for normalized version

$$\theta = \frac{r}{\sigma_1} = \left[\frac{q}{R_{11}(0,0)}\right]^{\frac{1}{2}} = \sqrt{\phi} \,, \tag{167}$$

namely

$$p_\theta(u) = 2\,u\,p_\phi(u^2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(u^2 + \eta_1^2\right)\right] \cosh(\eta_1 u) \quad \text{for } u > 0 \,. \tag{168}$$

This PDF is displayed in figure 2 for $\eta_1 = 0(.5)3$. At the origin this PDF is finite and nonzero, with value $(2/\pi)^{\frac{1}{2}} \exp(-\eta_1^2/2)$, indicating the possibility of occasional deep fades (unless η_1 is large). As parameter η_1 gets large, this PDF approaches Gaussian; in fact, directly from (168) and figure 2, we obtain

$$p_\theta(u) \sim (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(u - \eta_1\right)^2\right] \quad \text{for } \eta_1 > 3 \,. \tag{169}$$

The corresponding PDF for normalized power-scaling variate $\phi = q/R_{11}(0,0) = q/\sigma_1^2$ was given in (166) and is plotted in figure 3. It is seen to possess a substantial cusp at the origin, which behaves as $(2\pi u)^{-\frac{1}{2}} \exp(-\eta_1^2/2)$. This is the most severe case of deep fading that model (45) can yield, namely for $M = 1$.

When $M = 2$, that is, two components in fading model (45), the characteristic function in (163) reduces to

$$f_\phi(\xi) = \left(1 - i2\xi\right)^{-\frac{1}{2}} \left(1 - i2\xi\ r^{(2)}\right)^{-\frac{1}{2}} \exp\left[\frac{i\xi\ \eta_1^2}{1 - i2\xi} + \frac{i\xi\ \eta_2^2\ r^{(2)}}{1 - i2\xi\ r^{(2)}}\right]. \tag{170}$$

However, instead of Fourier transforming this result, the PDF of ϕ is best found by reverting to definition (45), namely $q = (c_1 + g_1)^2 + (c_2 + g_2)^2$, and performing the probability integrals directly in the g_1,g_2 plane. The result is

$$p_\phi(u) = \frac{1}{4\pi\left(r^{(2)}\right)^{\frac{1}{2}}} \int_{-\pi}^{\pi} dt\ \exp\left[-\frac{1}{2}\left(C^2 + S^2\right)\right] \quad \text{for } u > 0\ , \tag{171}$$

where auxiliary functions

$$C \equiv u^{\frac{1}{2}} \cos(t) - \eta_1\ , \quad S \equiv \left(\frac{u}{r^{(2)}}\right)^{\frac{1}{2}} \sin(t) - \eta_2\ . \tag{172}$$

As a special subcase here, for $M = 2$, let $r^{(2)} = 1$; that is, let both random components $\{g_m(t,f)\}$ have equal power. Then (170) yields characteristic function

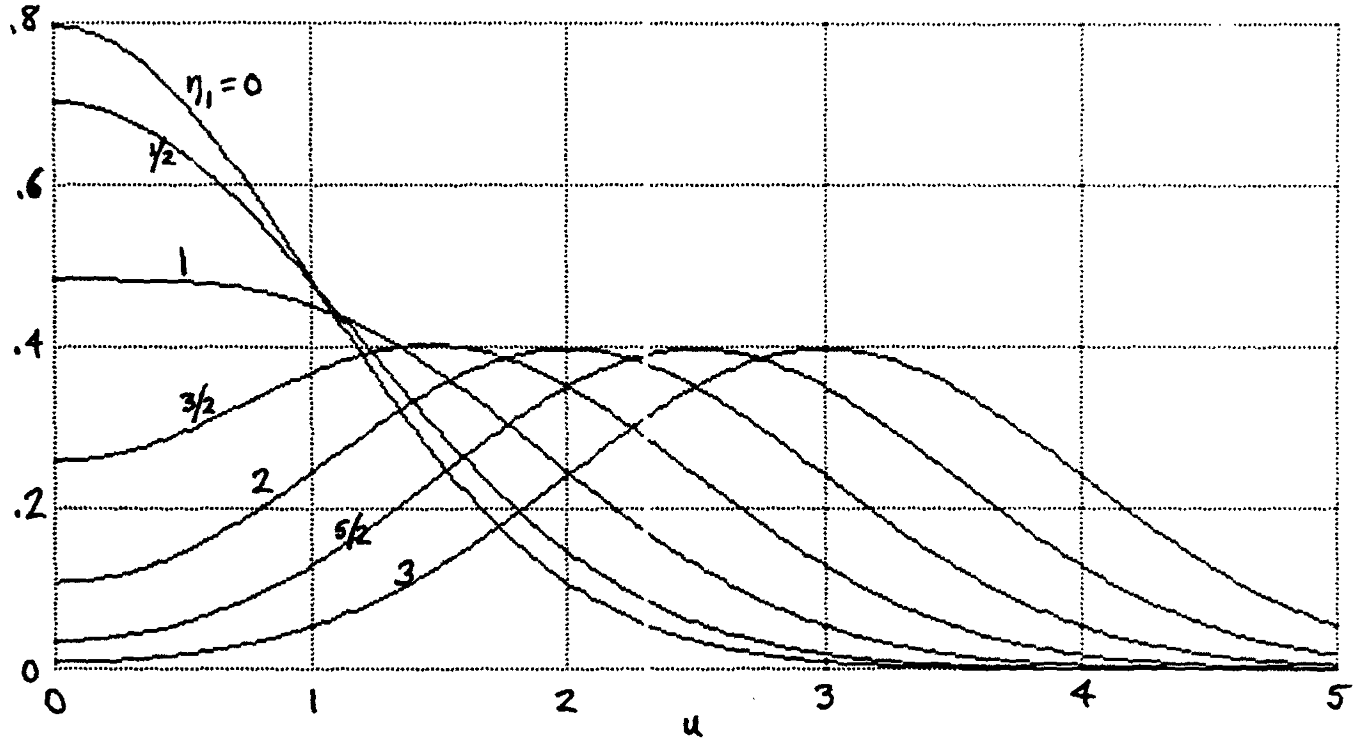

Figure 2. PDF of r/σ_1 for M = 1

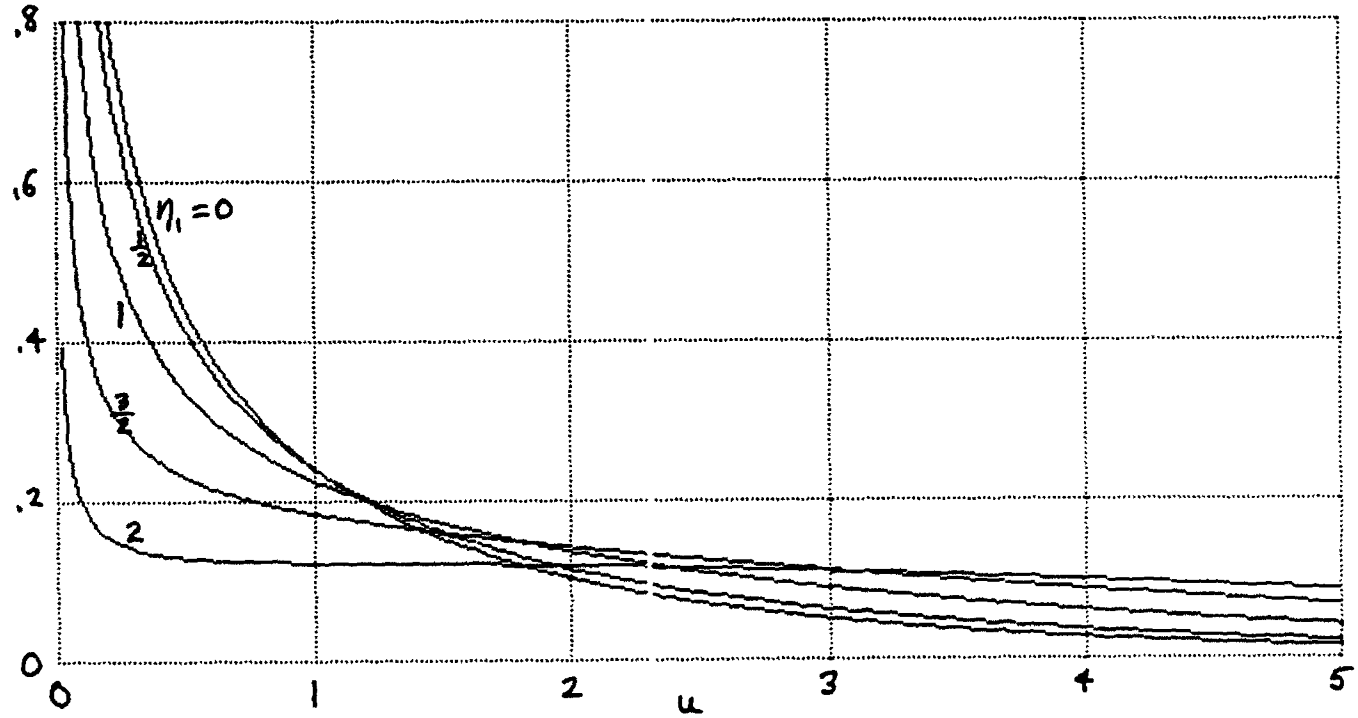

Figure 3. PDF of q/σ_1^2 for M = 1

$$f_\phi(\xi) = (1 - i2\xi)^{-1} \exp\left[\frac{i\xi\ \eta^2}{1 - i2\xi}\right] , \tag{173}$$

where constant

$$\eta^2 \equiv \eta_1^2 + \eta_2^2 = \frac{c_1^2 + c_2^2}{R_{11}(0,0)} \quad \text{for } M = 2 . \tag{174}$$

The corresponding probability density function for ϕ is

$$p_\phi(u) = \frac{1}{2}\, I_o(\eta\sqrt{u}) \exp\left[-\frac{1}{2}\left(u + \eta^2\right)\right] \quad \text{for } u > 0 . \tag{175}$$

That for $\theta = \sqrt{\phi}$ follows immediately as

$$p_\theta(u) = u\, I_o(\eta u) \exp\left[-\frac{1}{2}\left(u^2 + \eta^2\right)\right] \quad \text{for } u > 0 . \tag{176}$$

The PDF in (175) is displayed in figure 4. Now, the origin value

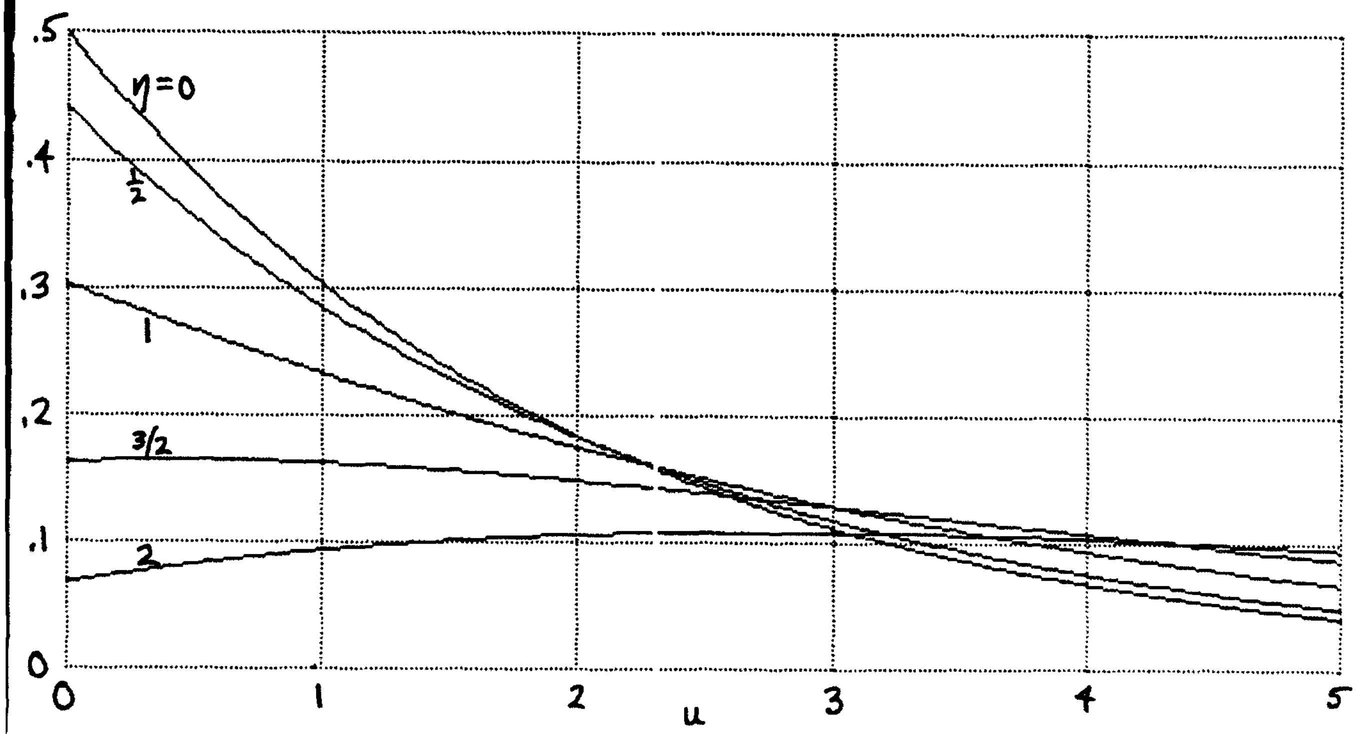

Figure 4. PDF of q/σ_1^2 for $M = 2$, $r^{(2)} = 1$

is finite, namely .5 $\exp(-\eta^2/2)$, in contrast with figure 3 for $M = 1$. The cusps that were in figure 3 are now absent for $M = 2$.

When $M = 2$ but $r^{(2)} \neq 1$, a closed form like (175) is not available for the PDF of ϕ. Instead, we must revert to the general relation in (171) - (172) and perform the numerical integration for each set of parameter values of interest. An example for $r^{(2)} = 1/2$ and $\eta_2 = \eta_1$ is given in figure 5, and a second example for $r^{(2)} = 1/2$ and $\eta_2 = \eta_1/2$ is given in figure 6. A common feature of all these densities is the finite nonzero values at the origin. That is, deep fades are still common for fading model (45) when $M = 2$.

For larger M and completely arbitrary parameters, it is necessary to numerically Fourier transform general characteristic function (163) in order to determine the PDF of ϕ. However, for the special case where

$$M \text{ is even} \quad \text{and} \quad r^{(m)} = 1 \quad \text{for } 1 \le m \le M , \tag{177}$$

then (163) reduces to

$$f_\phi(\xi) = (1 - i2\xi)^{-M/2} \exp\left[\frac{i\xi \ \eta^2}{1 - i2\xi}\right] , \tag{178}$$

where constant

$$\eta^2 = \sum_{m=1}^{M} \eta_m^2 . \tag{179}$$

The corresponding PDF of ϕ is then

$$p_\phi(u) = \frac{1}{2} \left(\frac{\sqrt{u}}{\eta}\right)^N I_N(\eta\sqrt{u}) \exp\left[-\frac{1}{2}\left(u + \eta^2\right)\right] \quad \text{for } u > 0 , \tag{180}$$

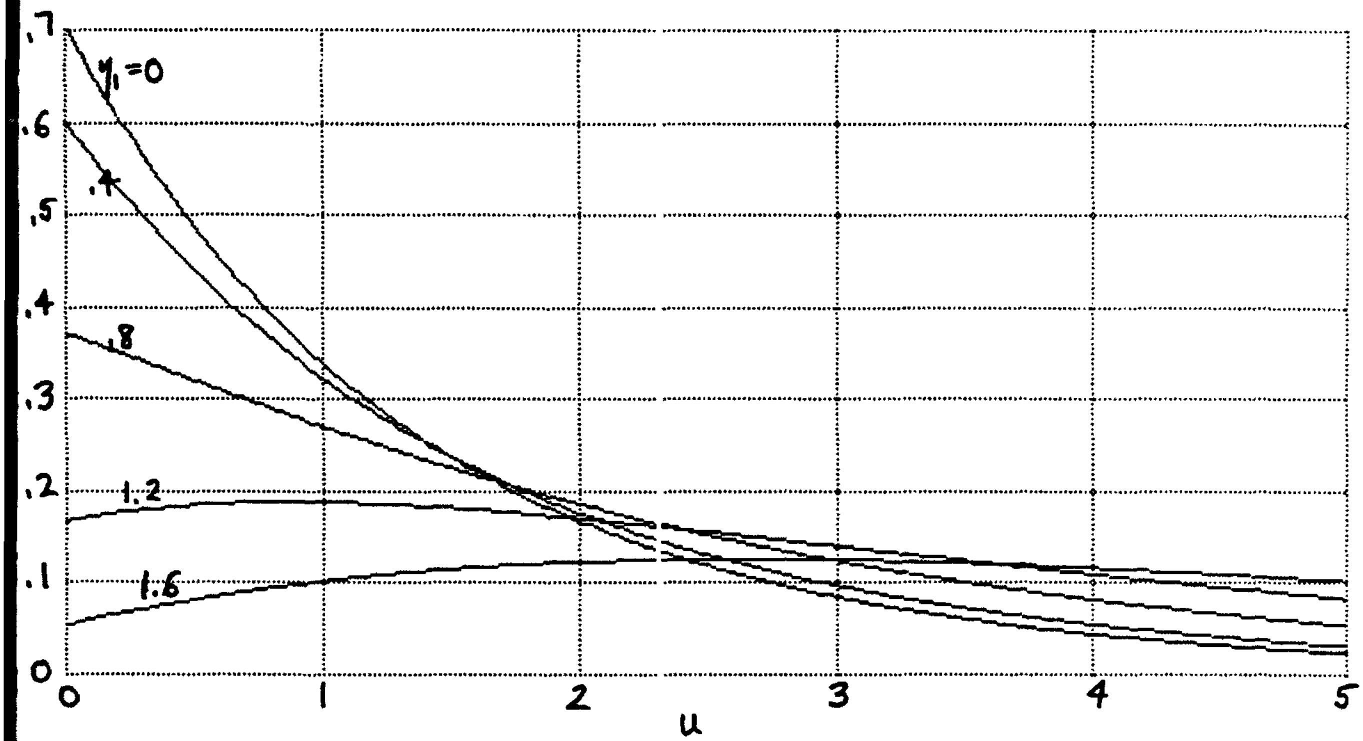

Figure 5. PDF of q/σ_1^2 for M = 2, $r^{(2)}$ = 1/2, $\eta_2 = \eta_1$

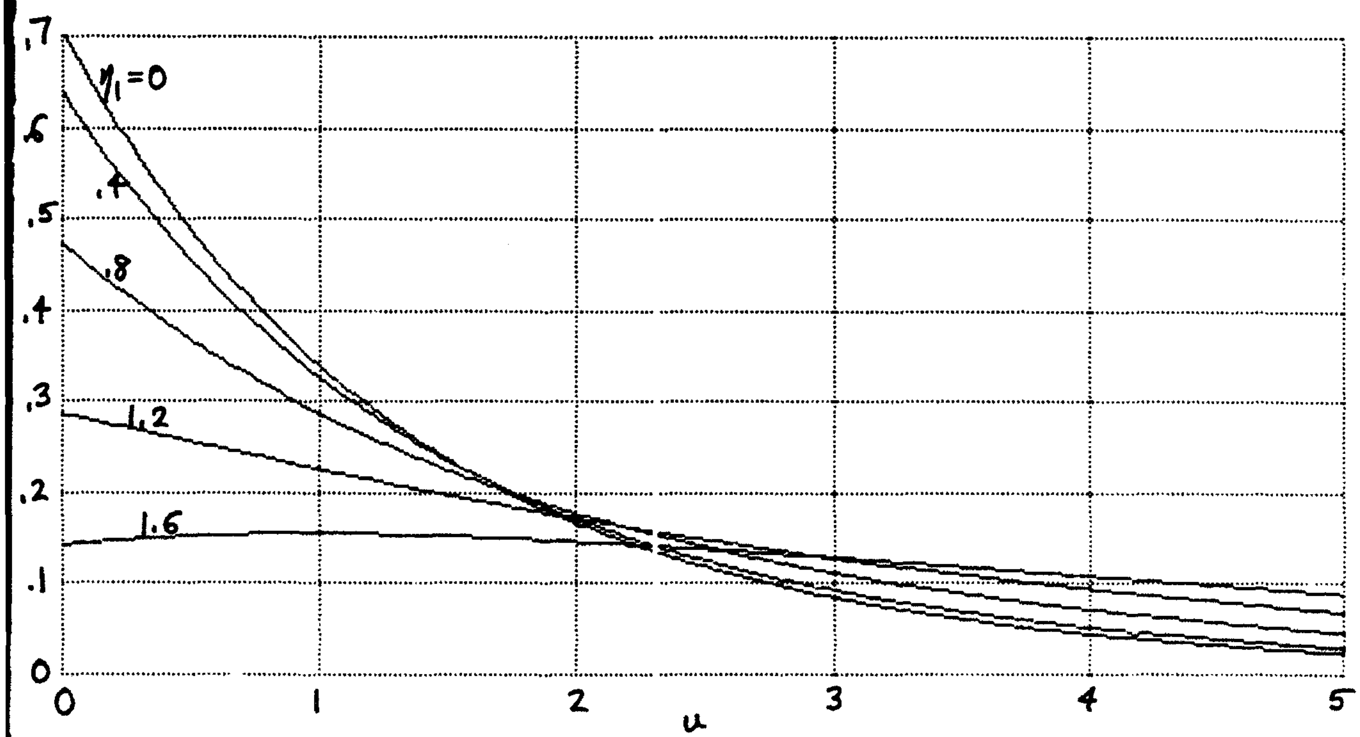

Figure 6. PDF of q/σ_1^2 for M = 2, $r^{(2)}$ = 1/2, $\eta_2 = \eta_1/2$

where $N = M/2 - 1$. This reduces to (175) for $M = 2$.

The PDF for $\theta = \sqrt{\phi}$ follows as given by the rule given in (168), namely

$$p_\theta(u) = u \left(\frac{u}{\eta}\right)^N I_N(\eta u) \exp\left[-\frac{1}{2}\left(u^2 + \eta^2\right)\right] \quad \text{for } u > 0 \ . \qquad (181)$$

This reduces to (176) for $M = 2$.

Two examples of PDF (180) for random variable ϕ are displayed in figure 7 for $M = 4$ and in figure 8 for $M = 6$, respectively. The former PDF is zero at the origin, while the latter is zero and has zero first derivative at the origin. Thus, deep fades are less likely for the larger values of M in fading model (45).

A general procedure for determining the probability density function $p_\phi(u)$ of random variable ϕ from characteristic function $f_\phi(\xi)$ in (163), for arbitrary M, $\{r^{(m)}\}$, $\{\eta_m\}$, is presented at the end of appendix D. It employs some efficient recursions for fast and accurate numerical evaluation of $p_\phi(u)$.

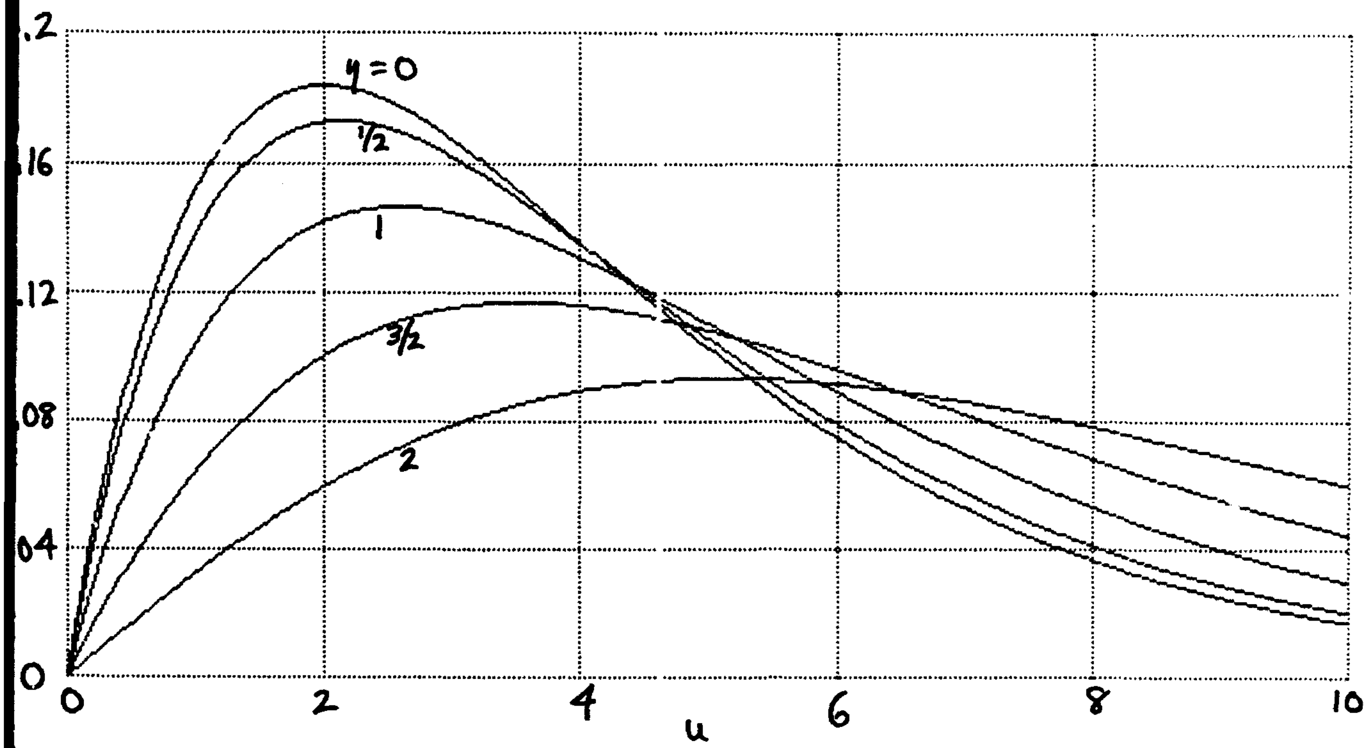

Figure 7. PDF of q/σ_1^2 for M = 4, $r^{(m)}$ = 1 for all m

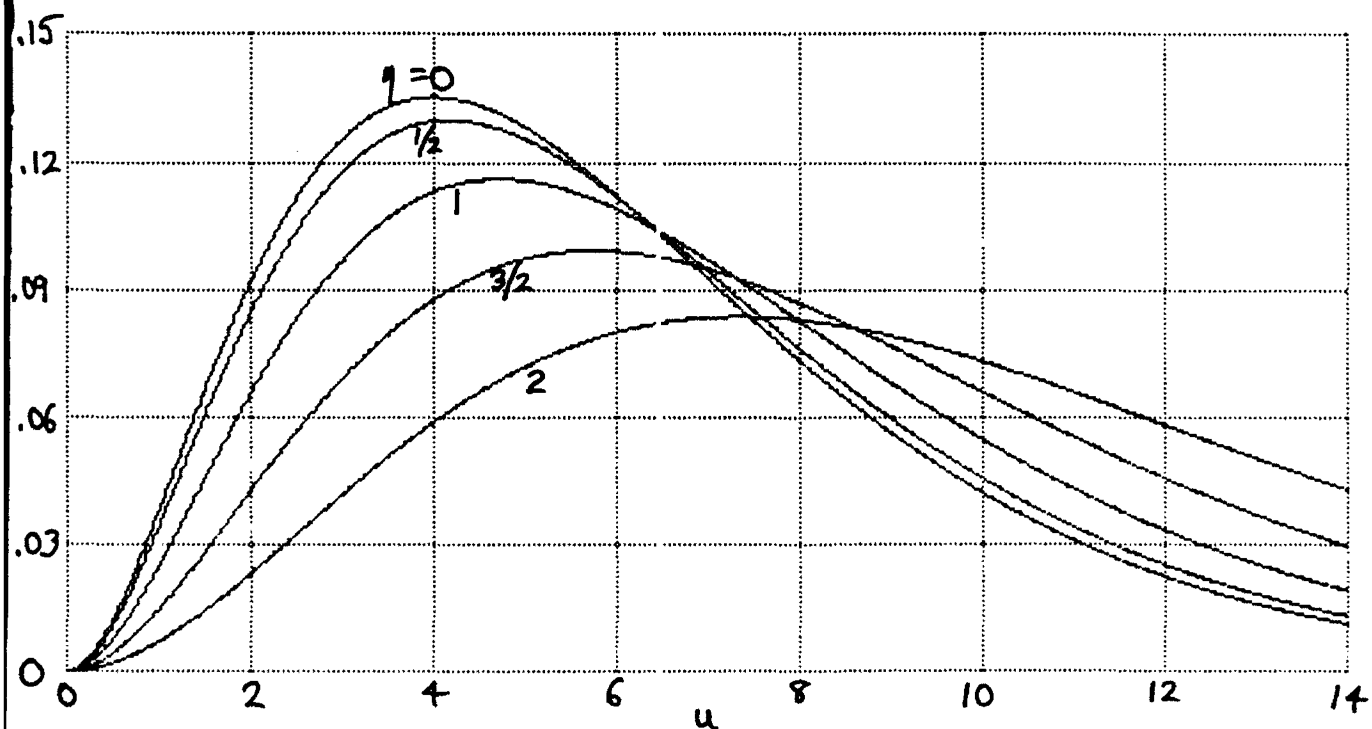

Figure 8. PDF of q/σ_1^2 for M = 6, $r^{(m)}$ = 1 for all m

PROBABILITY DISTRIBUTION OF PROCESSOR OUTPUT γ

The characteristic functions $f_\gamma(\xi)$ of processor output γ, under a variety of special cases, were derived in an earlier section; in particular, see (124) - (160). Although these are compact closed forms, they can encounter computational problems if we attempt to directly utilize Fourier transforms to determine the corresponding probability density functions and exceedance distribution functions. In particular, since the characteristic functions decay only as ξ^{-K} as $\xi \to \infty$, truncation error can become a significant problem, especially for small K, the number of signal pulses.

EXPANSION FOR DISTRIBUTION FUNCTION

In appendix D, series expansions for $f_\gamma(\xi)$ and the corresponding probability density function $p_\gamma(u)$ are derived in a form which involves <u>only</u> positive expansion coefficients and terms. Furthermore, efficient and accurate recursions are developed for the evaluation of the coefficients, the probability density function, and the exceedance distribution function.

As an example of the general procedure, the results for general characteristic functions (136) and (158) are summarized here. Define respectively, for $1 \le m \le M$, $1 \le k \le K$,

$$T_{mk} = \frac{2E_{1m}}{N_o}\,\underline{\lambda}_k^{(m)} \quad \text{or} \quad T_{mk} = \frac{2E_{1m}}{N_o}\,\underline{\lambda}_k\ , \qquad e_{mk} = \underline{\varepsilon}_k^{(m)^2}\ . \qquad (182)$$

(136) (158)

Then, the exceedance distribution function of processor output γ is given by (D-16) as

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p \, H_{K+p-1}\left(\frac{u}{2}\right) \quad \text{for } u > 0 , \tag{183}$$

where the quantities required are found in the following fashion.

$$F = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 + T_{mk}\right)\right]^{-\frac{1}{2}} , \tag{184}$$

$$\alpha_o = -\frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 + T_{mk}} ,$$

$$\alpha_p = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{T_{mk}^{p-1}}{(1 + T_{mk})^p} \left[\frac{T_{mk}}{p} + \frac{e_{mk}}{1 + T_{mk}}\right] \quad \text{for } p \geq 1 , \tag{185}$$

$$g_o = \exp(\alpha_o) , \qquad g_p = \frac{1}{p} \sum_{n=1}^{p} n \, \alpha_n \, g_{p-n} \quad \text{for } p \geq 1 , \tag{186}$$

$$H_n(x) = \exp(-x) \sum_{k=0}^{n} \frac{x^k}{k!} \quad \text{for } n \geq 0 , \; x \geq 0 . \tag{187}$$

Since the eigenvalues of a covariance matrix can never be negative, it is seen from (185) that all the $\{\alpha_p\}$ are nonnegative, with the exception of α_o. However, α_o is used only once in (186) to generate a positive g_o, while the recursion in (186) for $\{g_p\}$ utilizes only nonnegative quantities. Thus, all the coefficients $\{g_p\}$ and scale factor F in (183) are nonnegative. The function $H_n(x)$ in (187) is obviously positive

for $x \geq 0$, thereby guaranteeing that no negative terms will appear in expansion (183). Furthermore, there is a very efficient recursion for the $\{H_n(x)\}$; see (D-18). A sample program for the evaluation of a modified form of (183) is given in appendix D.

In the absence of signal, all the $\{T_{mk}\}$ and $\{e_{mk}\}$ in (182) are zero. Then, all the $\{\alpha_p\}$ and $\{g_p\}$ become zero except for g_o which is 1. Expansion (183) for the exceedance distribution function of processor output γ then reduces to simply the one term $H_{K-1}(u/2)$, which is consistent with the noise-only characteristic function $(1 - i2\xi)^{-K}$ for processor output γ; see (42), for example. That is, the false alarm probability is

$$P_F = H_{K-1}\left(\frac{u}{2}\right) . \tag{188}$$

An error bound for sum (183), terminated at the $p = N$ term, is given in (D-20). Also, the modifications required to treat the slightly different forms of characteristic functions encountered in (124) and (144) are presented in (D-23) - (D-26). Finally, a refinement of the expansion procedure in (183), which is more rapidly convergent and useful for larger signal-to-noise ratios, is given in (D-27) - (D-36).

FUNDAMENTAL INPUTS AND COMPUTATIONAL PROCEDURE

In the remainder of this section, we will restrict attention to characteristic function (158) which pertains to special case 6, namely uncorrelated components with proportional covariances; see (67). However, before we list the fundamental inputs that are required to conduct the numerical evaluation of the exceedance distribution function of processor output γ in this case, we make an additional modification for later convenience in plotting and comparison.

From (52), (38), and (67), we have

$$E_{km} = E_{1m} = \tilde{E}\, R_{11}(0,0)\, r^{(m)} \; . \tag{189}$$

Then, using (53), the average random received signal energy in one pulse is

$$E_1 = \sum_{m=1}^{M} E_{1m} = \tilde{E}\, R_{11}(0,0) \sum_{m=1}^{M} r^{(m)} \; . \tag{190}$$

This enables us to express the signal-to-noise ratio measure as

$$\frac{E_{1m}}{N_o} = \frac{E_1}{N_o}\, \psi_m \; , \quad \text{where} \quad \psi_m \equiv \frac{r^{(m)}}{\sum_{m=1}^{M} r^{(m)}} \quad \text{for } 1 \le m \le M \; . \tag{191}$$

The quantities $\{\psi_m\}$ represent the fractional strengths of each of the M random components $\{g_m(t,f)\}$ in fading model (45).

The fundamental inputs required to evaluate the exceedance distribution function of processor output γ are now

K, number of signal pulses in figure 1,

M, number of fading components in (45),

E_1/N_o, signal energy to noise density ratio,

$\{r^{(m)}\}$ for $1 \le m \le M$; see (67),

$\{D_{km}/N_o\}$ for $1 \le k \le K$, $1 \le m \le M$; see (50),

$R_{11}(\tau,\nu)/R_{11}(0,0)$; see (47),

$\{t_k\}$ and $\{f_k\}$ for $1 \le k \le K$; see (44). (192)

The quantity E_1/N_o is a measure of the receiver input average random signal-to-noise ratio for one signal pulse, while $r^{(m)}$ is the relative strength of the m-th fading component. Ratio D_{km}/N_o is a measure of the receiver input deterministic signal-to-noise ratio for the k-th signal pulse and m-th component. The function $R_{11}(\tau,\nu)/R_{11}(0,0)$ is the normalized fading covariance function for the medium at time separation τ and frequency separation ν. Parameters t_k and f_k are the time and frequency locations, respectively, of the k-th signal pulse in time,frequency space.

The first quantity that must be computed is the normalized K×K covariance matrix $\underline{C}$ given by (140). Then, its eigenvalue matrix $\underline{\Lambda}$ and normalized modal matrix Q are found by solving characteristic-value matrix equation (141). This yields eigenvalues $\{\underline{\lambda}_k\}$ and eigenvectors $\{V_k\}$ for $1 \le k \le K$. The components of column vector V_k are $v_{k1}, \cdots, v_{kK}$; see (102). We then compute $\underline{\varepsilon}_k^{(m)}$ from (142) or (157) for $1 \le k \le K$, $1 \le m \le M$. At this point, the parameters $\{T_{mk}\}$ and $\{e_{mk}\}$ in (182) can be computed for $1 \le k \le K$, $1 \le m \le M$, and the procedure in (183) - (187) can be employed. In particular, (182) and (191) yield

$$T_{mk} = \frac{2E_1}{N_o} \psi_m \underline{\lambda}_k \quad \text{for} \quad 1 \le m \le M \, , \quad 1 \le k \le K \, . \tag{193}$$

NUMERICAL PERFORMANCE RESULTS

The first set of examples are selected to enable a comparison with the approximate procedure and results in [12]. In figure 9, below, we plot the required per-pulse input SNR (signal-to-noise ratio) measure E_1/N_o (in dB) versus the number of signal pulses K, for values of the adjacent-pulse normalized (amplitude) covariance, Cov_1, equal to 0, .5, $\sqrt{.5}$, and 1. In particular, the covariance $R_{11}(\tau,\nu)$ was taken as exponential in τ, and the signal pulses have no frequency shifts $\{f_k\}$ and are equally spaced in time locations $\{t_k\}$; that is, $Cov_1 = R_{11}(t_{k+1}-t_k,0)/R_{11}(0,0)$ for all k. Also, the KM deterministic signal-to-noise ratios $\{D_{km}/N_o\}$ are all zero, and the M strengths $\{r^{(m)}\}$ are all equal to 1; this duplicates the situation in [12]. The particular case in figure 9 pertains to false alarm probability P_F = 1E-6, detection probability P_D = .5, and M = 1 fading component; this last choice corresponds to parameter m = 1/2 in [12]. The results here in figure 9 for Cov_1 = 0 and Cov_1 = 1 agree precisely with ρ = 0 and ρ = 1 in [12; figure 11], as expected, while the results for $Cov_1 = \sqrt{.5}$ are in rather good agreement with those for ρ = .5. This selection of parameters reflects the property that ρ in [12] is Cov_1^2 here; see the section entitled Main Program in appendix D for additional details.

The only change in figure 10 is to require a larger detection probability, namely P_D = .9. Although the Cov_1 = 0 and 1 results

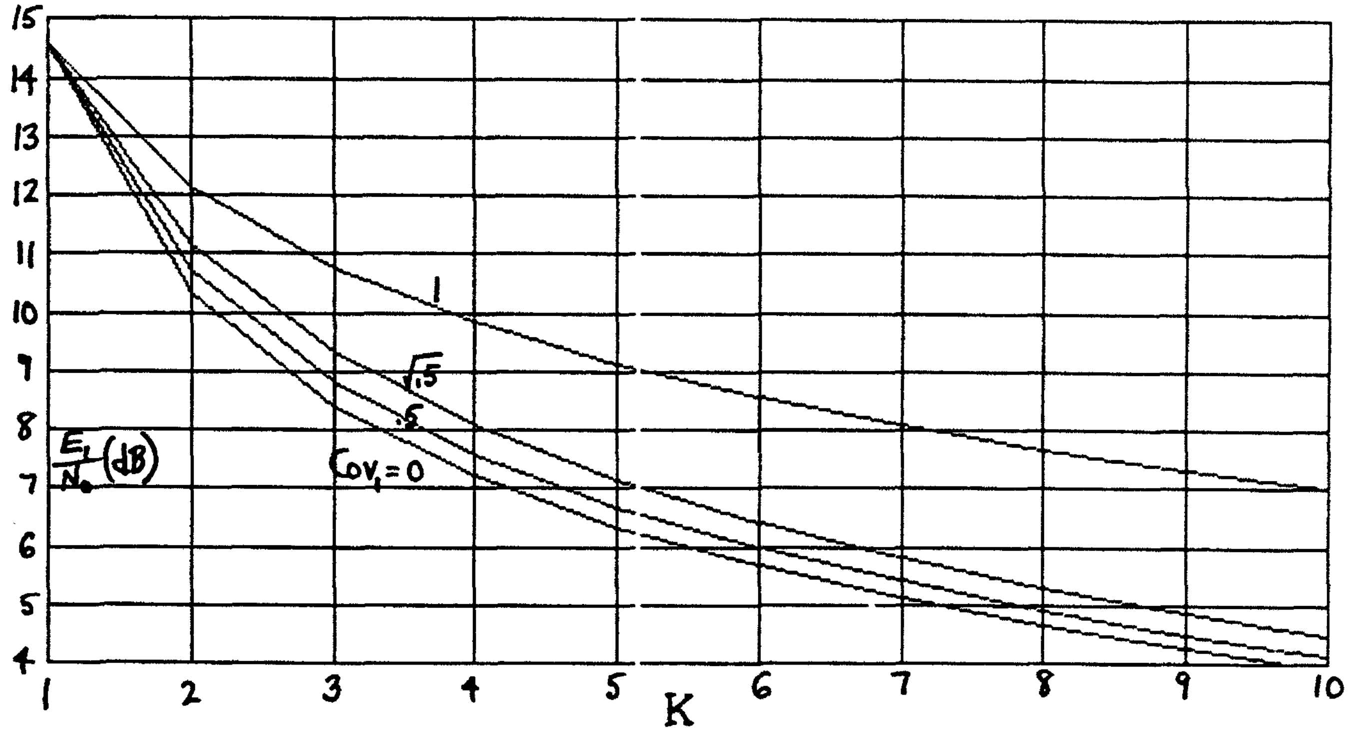

Figure 9. Required SNR for P_F = 1E-6, P_D = .5, M = 1

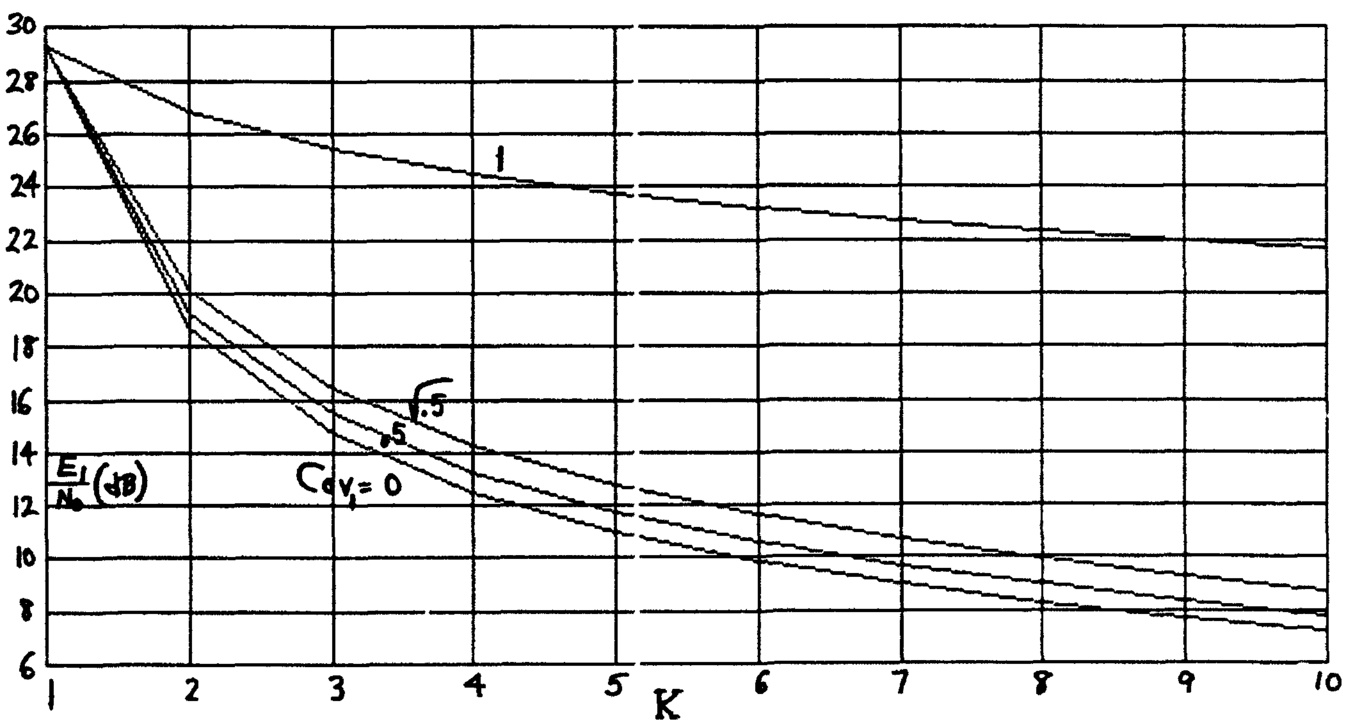

Figure 10. Required SNR for P_F = 1E-6, P_D = .9, M = 1

agree with [12; figure 12], the exact results here for $Cov_1 = \sqrt{.5}$ are markedly different from the approximate results for $\rho = .5$ in the earlier work. For example, 2.6 dB less is required for K = 2 and K = 3, while 2.3 dB less is required for K = 4. That is, the earlier approximation in [12] was somewhat pessimistic in its performance predictions.

We must observe from figure 10 that the effect of correlated fading is not overly significant until the normalized covariance approaches 1. For example, for K = 2, the cost of Cov_1 increasing from 0 to $\sqrt{.5}$ is 1.4 dB, while the cost of Cov_1 going from $\sqrt{.5}$ to 1 is an additional 6.7 dB. The distinction is even greater for K = 10, with the same comparison requiring 1.6 dB versus an additional 12.9 dB.

The example in figure 11 pertains to M = 2 fading components, (which corresponds to m = 1 in [12]); all other parameters are the same as figure 9 above. These results can be compared directly with [12; figure 7]; they reveal identical performance for $Cov_1 = 0$ and 1, and rather good agreement for $Cov_1 = \sqrt{.5}$ (versus $\rho = .5$).

When the detection probability P_D is increased to .9 in figure 12, these exact results reveal that lower values of E_1/N_o are required than the approximation in [12; figure 8] predicted, at least for $Cov_1 = \sqrt{.5}$ ($\rho = .5$). Also, as was seen in figure 10, the cost of normalized covariance Cov_1 approaching 1 is very significant in terms of increased signal level; that is, the $Cov_1 = \sqrt{.5}$ curves are well below the $Cov_1 = 1$ curves in figures

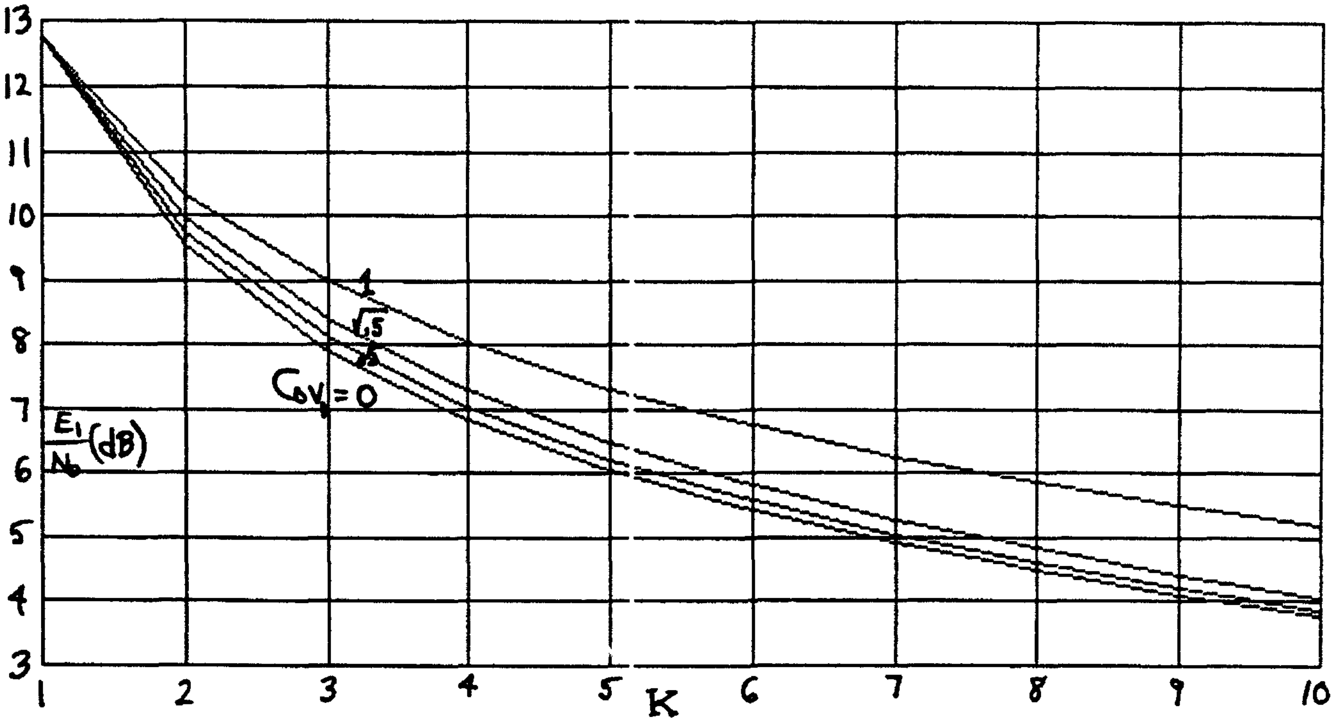

Figure 11. Required SNR for P_F = 1E-6, P_D = .5, M = 2

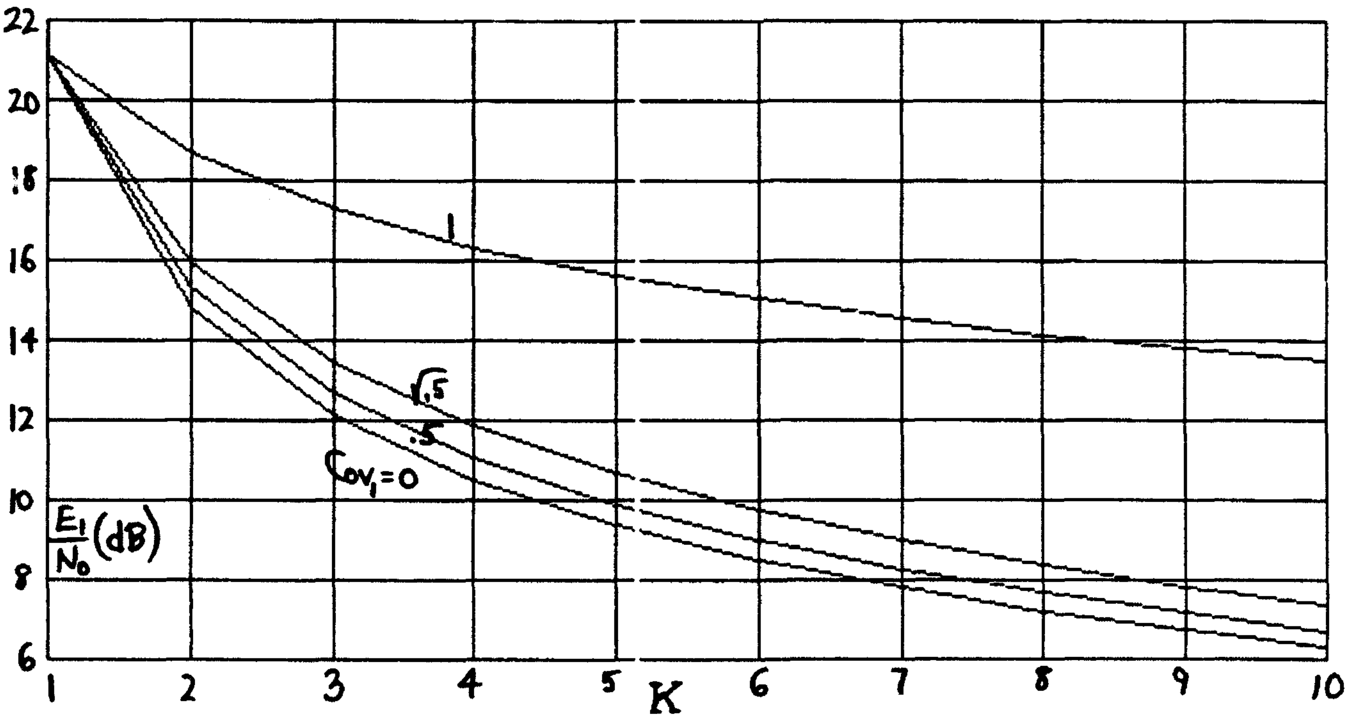

Figure 12. Required SNR for P_F = 1E-6, P_D = .9, M = 2

10 and 12, especially for the larger K values.

The detrimental effect of highly correlated fading pulses is studied quantitatively in figure 13, where normalized covariance Cov_1 is varied from 0 to 1 as the other various parameters are kept fixed. The number K of signal pulses is kept at 2. These plots reveal that for detection probability $P_D = .5$, increased Cov_1 does not lead to a significantly large increase in required input signal level. However, for the better detection probability of $P_D = .9$, higher covariances can be very damaging, requiring additional signal strength to maintain the desired level of performance. For example, as Cov_1 increases from .9 to 1 in the uppermost example in figure 13, the signal must be increased by 4.9 dB. (The kinks in the curves are due to discretization of the abscissa at increment .1 for Cov_1.)

When K is increased to 6, the results in figure 14 reveal this effect in a more pronounced fashion. An additional 9 dB is now required when Cov_1 is increased from .9 to 1 for the upper curve.

All of the above results have had deterministic signal-to-noise ratio measures $\{D_{km}/N_o\}$ equal to zero. The program listed in appendix D has the capability of incorporating arbitrary values for these parameters as well as others, such as $\{r^{(m)}\}$. An example where all the potential is exercised, and all parameters have nonzero values, is displayed in figure 15. Here, the detection probability P_D is varied from .5 to .999 and the required E_1/N_o is calculated (in dB). It is seen that a sharp

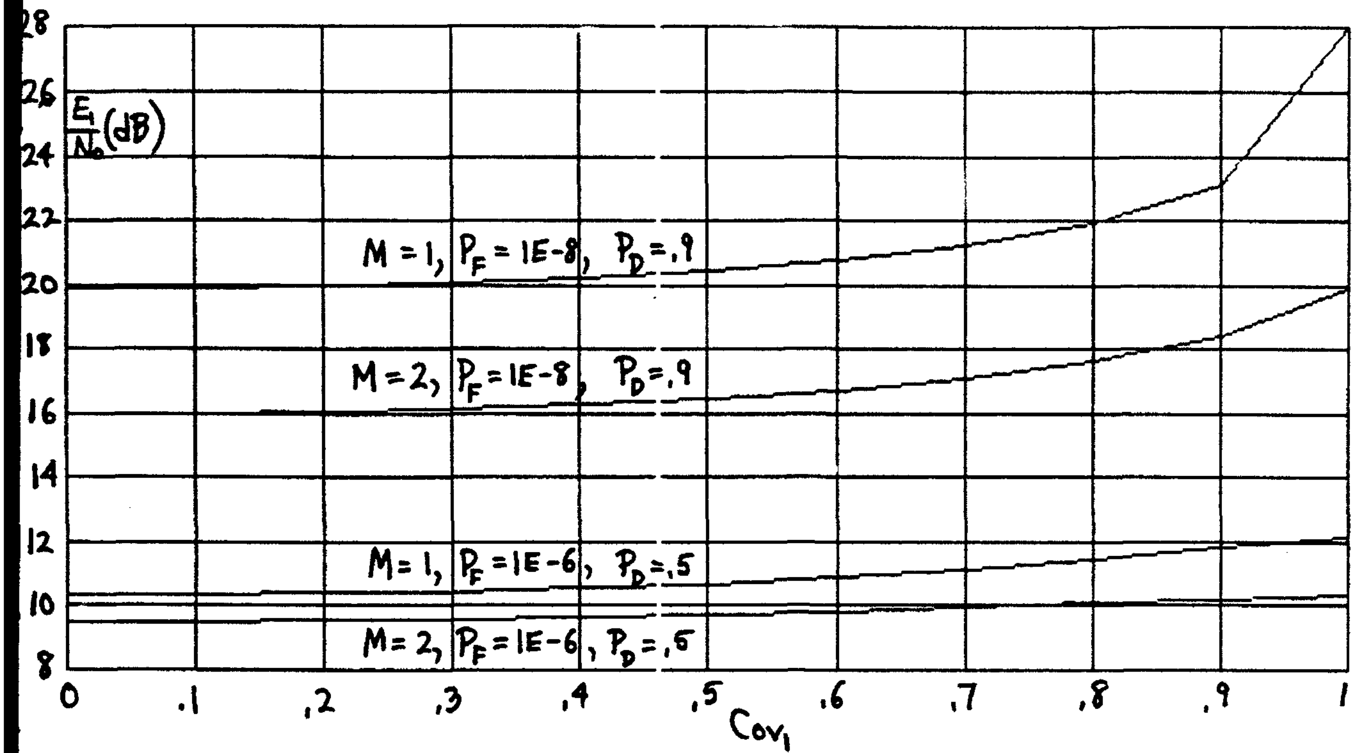

Figure 13. Required SNR versus Cov_1, K = 2

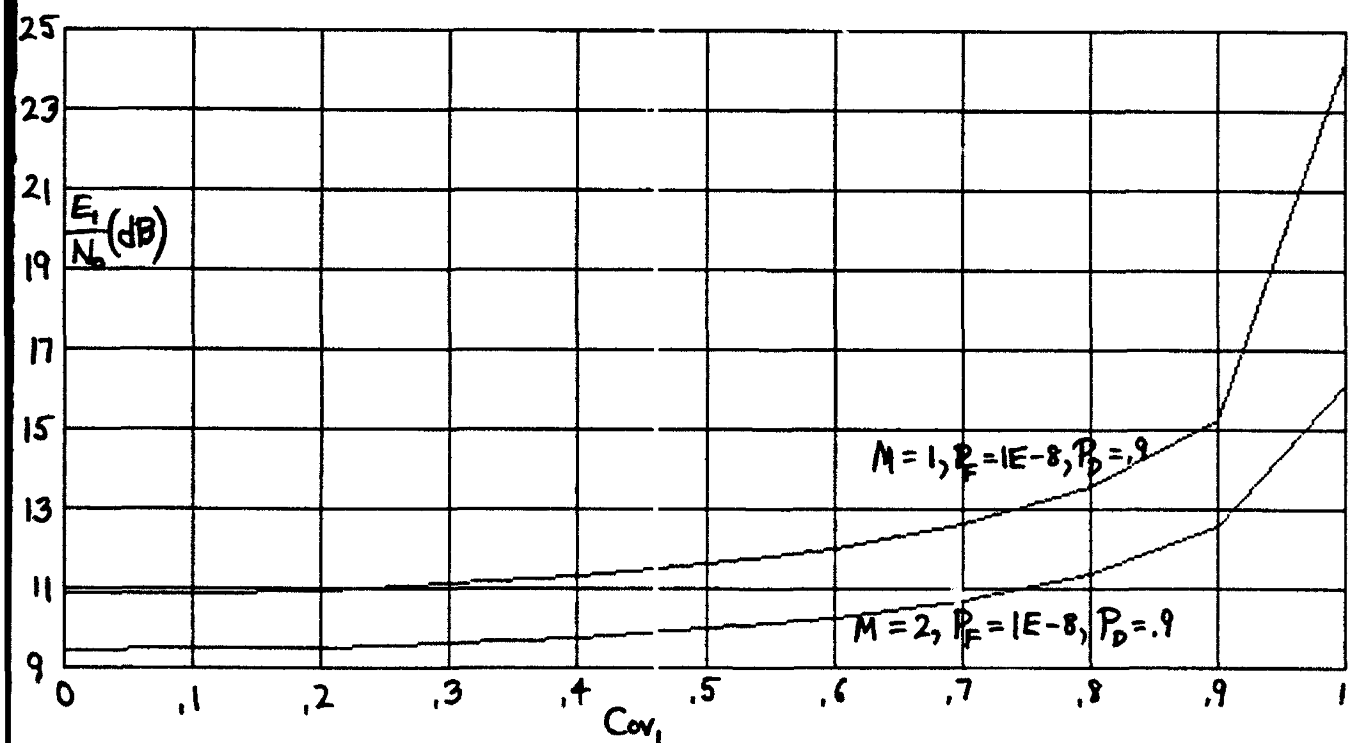

Figure 14. Required SNR versus Cov_1, K = 6

increase is observed as P_D increases above .9, eventually tending to ∞ as $P_D \to 1$. The particular parameter values are listed below:

$$P_F = 1E\text{-}6, \quad K = 4, \quad M = 2,$$

$$r^{(m)} = 1 \quad \text{for } 1 \le m \le M,$$

$$D_{km}/N_o = 1 \text{ for } 1 \le k \le K,\ 1 \le m \le M,$$

$$\{t_k\} = 1,2,3,4, \qquad \{f_k\} = 1,4,2,3,$$

$$\mathrm{Cov}(\tau,\nu) = \frac{R_{11}(\tau,\nu)}{R_{11}(0,0)} = \exp\left(-\frac{\tau^2}{11} - \frac{\nu^2}{10}\right) \quad . \tag{194}$$

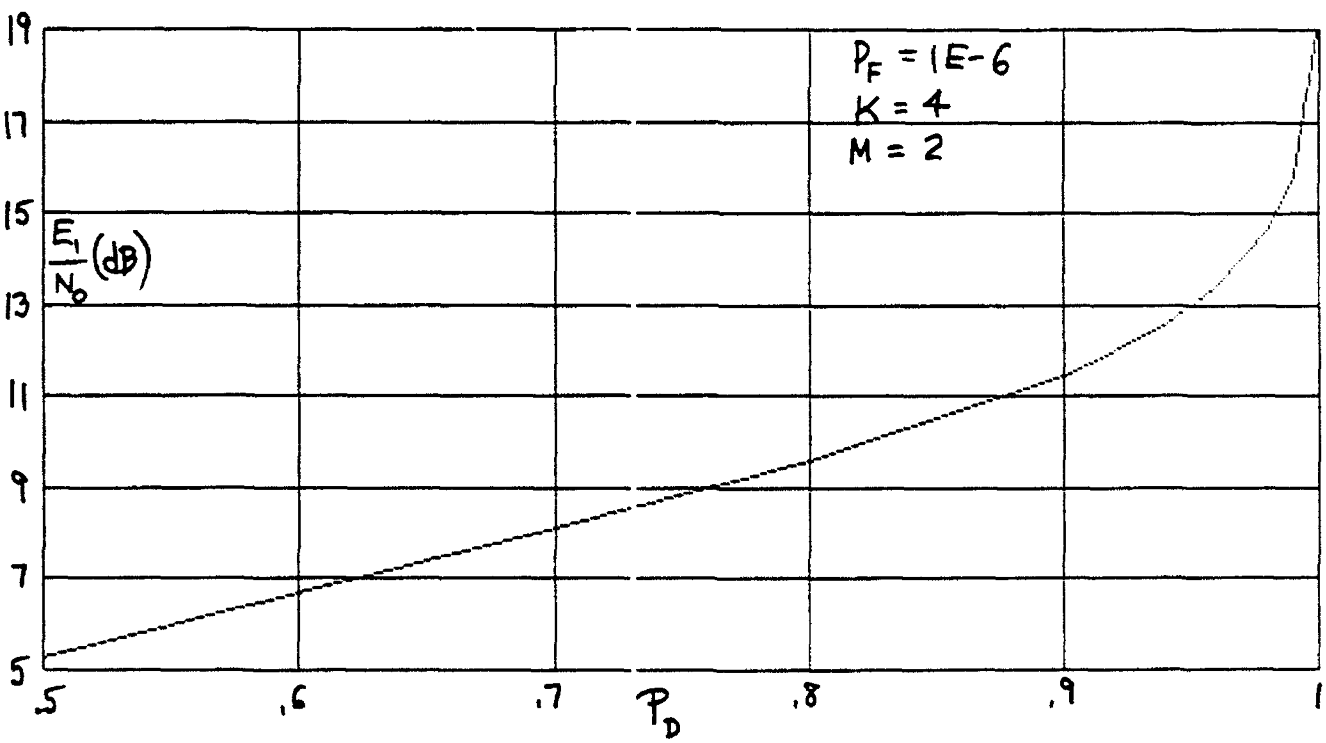

Figure 15. Required SNR versus P_D

SUMMARY

The characteristic function of processor output γ in figure 1 has been derived in closed form for a wide variety of fading conditions and signal formats. The corresponding probability density functions and exceedance distribution functions of γ have then been expanded in convergent series, by means of a novel expansion technique which make efficient use of recursions for rapid and accurate evaluation. These series typically require on the order of 30 to 60 terms for ten decimal accuracy. A program is given which incorporates all these features, including allowance for both deterministic and random components of arbitrary strengths in the fading medium.

One of the most useful results pointed out by this study is that the degree of correlation between the fading signal pulses can be fairly significant without suffering great degradations. That is, when the normalized covariance approaches 1, meaning that all signal pulses tend to fade together, the performance losses are potentially large; however, for coefficients below .5, the losses are not too significant.

Comparison of these exact results with an earlier approximate procedure [12] reveals that the earlier approach generally gives pessimistic predictions of performance when the normalized covariance of the fading is intermediate between 0 and 1. In some cases, the discrepancy can be several dB. This result indicates and emphasizes the need for accurate treatment of systems which must perform well, that is, yield high detection

probabilities in fading media. An extension of this work to a fading medium in which the background noise level is unknown and must be estimated from a finite sample in a noise-only region of time,frequency space, is currently underway by this author and will be reported on shortly.

The major result utilized here is the characteristic function of a quadratic form in correlated nonzero-mean Gaussian random variables; this result is presented in appendix B. It relies on the ability to solve the generalized eigenvalue problem; this latter procedure and solution is presented in appendix E. A related problem involving a slightly more general bilinear form is treated in appendix F. Finally, the characteristic function of the most general complex form with both first-order and second-order terms is solved in appendix G. These results are not utilized in this technical report but are presented for completeness and for possible future use and reference.

APPENDIX A. ANALYTIC AND COMPLEX ENVELOPE PROCESSES

Let $n(t)$ be a stationary zero-mean real random process with covariance $R_n(\tau)$ and double-sided spectrum $G_n(f)$:

$$R_n(\tau) = \overline{n(t)\ n(t-\tau)}\ , \qquad G_n(f) = \int d\tau\ \exp(-i2\pi f\tau)\ R_n(\tau)\ . \tag{A-1}$$

The analytic process $n_+(t)$ is generated by eliminating the negative frequencies in $n(t)$ and by doubling its positive frequency contributions; that is, $n(t)$ is passed through a filter with transfer function $2\ U(f)$, where U is the unit step function.

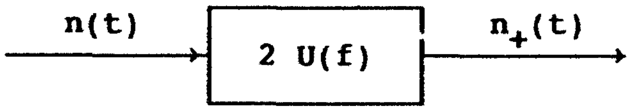

Figure A-1. Generation of Analytic Process

The analytic process can be expressed in terms of its real and imaginary parts according to

$$n_+(t) = n(t) + i\ n_H(t)\ , \tag{A-2}$$

where $n_H(t)$ is the Hilbert transform of $n(t)$. The complex envelope process $\underline{n}(t)$ is obtained by frequency down-shifting analytic process $n_+(t)$, in order to center its one-sided spectrum about $f = 0$. Thus, if f_o is a representative center frequency of $n_+(t)$, then

$$\underline{n}(t) = n_+(t)\ \exp(-i2\pi f_o t)\ . \tag{A-3}$$

The original real process n(t) then follows from (A-2) and (A-3):

$$n(t) = \mathrm{Re}\left[n_+(t)\right] = \mathrm{Re}\left[\underline{n}(t)\ \exp(i2\pi f_o t)\right] . \tag{A-4}$$

The spectrum of $n_+(t)$ follows from figure A-1 as

$$G_{n_+}(f) = 4\ U(f)\ G_n(f) , \tag{A-5}$$

while the spectrum of $\underline{n}(t)$ is obtained from (A-3) according to

$$G_{\underline{n}}(f) = G_{n_+}(f+f_o) = 4\ U(f+f_o)\ G_n(f+f_o) . \tag{A-6}$$

The covariances corresponding to (A-5) and (A-6) are

$$\overline{n_+(t)\ n_+^*(t-\tau)} \quad \text{and} \quad \overline{\underline{n}(t)\ \underline{n}^*(t-\tau)} , \tag{A-7}$$

respectively. The two complementary covariances are zero; that is,

$$\overline{n_+(t)\ n_+(t-\tau)} = 0 , \quad \overline{\underline{n}(t)\ \underline{n}(t-\tau)} = 0 . \tag{A-8}$$

These results follow from the fact that transfer function 2 U(f) in figure A-1 is single-sided; that is, $U(f) = 0$ for $f < 0$.

A situation that frequently arises in practice is where n(t) is a noise process with a spectrum $G_n(f)$ that is flat in a broad band of width W in the neighborhood of f_o, which essentially covers the signals and filters of interest. The spectra of the various processes are illustrated in figure A-2, where N_d is the double-sided noise spectral density level of n(t) in the neighborhood of $\pm f_o$. The spectrum $G_{\underline{n}}(f)$ of the complex envelope $\underline{n}(t)$ is flat in the neighborhood of $f = 0$ and has level $4\ N_d$ watts/Hz. Therefore, its covariance

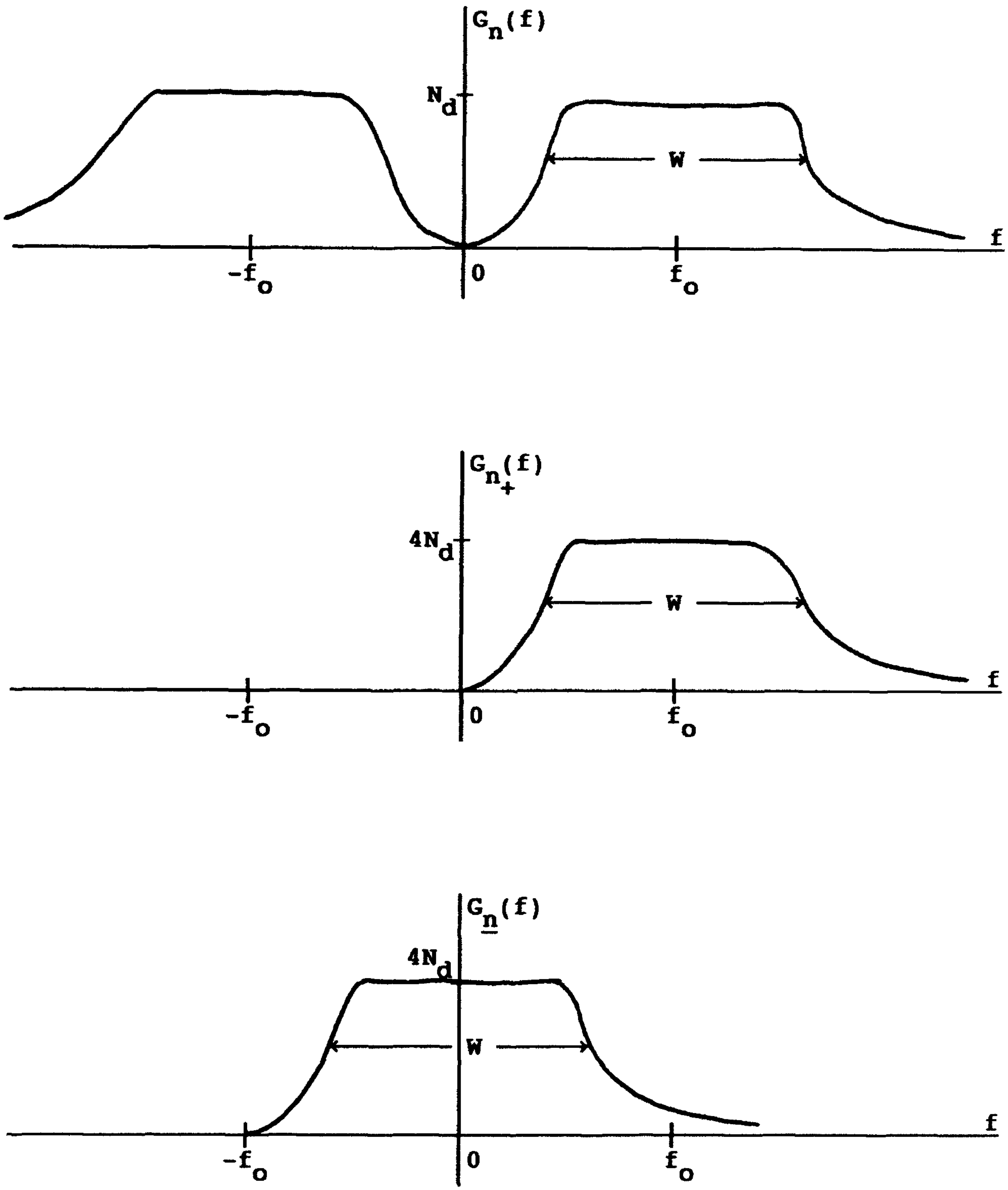

Figure A-2. Spectra of Analytic and Complex Envelope Processes

$$R_{\underline{n}}(\tau) = \int df \exp(i2\pi f\tau)\ G_{\underline{n}}(f) \tag{A-9}$$

is (relative to the signal and filter time functions) a sharp pulse centered at $\tau = 0$, with area $G_{\underline{n}}(0) = 4\ N_d$. Then, a good approximation for many purposes is to say that

$$R_{\underline{n}}(\tau) \cong 4\ N_d\ \delta(\tau)\ . \tag{A-10}$$

There is no need to let W and f_o tend to infinity in order to utilize this result; it only requires that W be somewhat larger than the bandwidths of the signals and filters of interest.

In terms of the one-sided noise spectral density level of n(t), that is, $N_o = 2\ N_d$, approximation (A-10) becomes

$$R_{\underline{n}}(\tau) \cong 2\ N_o\ \delta(\tau)\ . \tag{A-11}$$

This result may be compared with [19; page 48, (3.10) - (3.1 and page 72, (6.21) - (6.22)], where his N is our N_o.

It should be noted that original covariance $R_n(\tau)$ in (A-1) <u>cannot</u> be recovered from this approximation. That is, via (A-4),

$$R_n(\tau) = \frac{1}{2}\,\mathrm{Re}\left(R_{\underline{n}}(\tau)\ \exp(i2\pi f_o\tau)\right) = \tag{A-12}$$

$$\overset{?}{\cong} \frac{1}{2}\,\mathrm{Re}\left(4\ N_d\ \delta(\tau)\ \exp(i2\pi f_o\tau)\right) = 2\ N_d\ \delta(\tau)\ , \tag{A-13}$$

which is incorrect. The flaw is that the use of $\delta(\tau)$, which is tantamount to $W \to \infty$, must be accompanied by having let $f_o \to \infty$, in addition. That is, $\exp(i2\pi f_o\tau)$ in (A-12) must vary as fast as $R_{\underline{n}}(\tau)$ in order for the narrowband representation to be valid. The correct end result for $f_o = W/2$ is $R_n(\tau) \to N_d\ \delta(\tau)$ as $W \to \infty$.

APPENDIX B. CHARACTERISTIC FUNCTION OF QUADRATIC AND LINEAR FORM IN CORRELATED NONZERO-MEAN GAUSSIAN RANDOM VARIABLES

The N×1 real vector X is composed of correlated Gaussian components with mean vector E and covariance matrix C; that is,

$$\overline{X} = E \ , \quad \mathrm{Cov}(X) = \overline{(X - E)(X - E)^T} = C \ . \tag{B-1}$$

Covariance matrix C is N×N, real, symmetric, and nonnegative definite. Real N×1 vector E is completely arbitrary.

We shall be interested in obtaining the characteristic function of the quadratic and linear real form in X given by

$$q = q(X) = X^T \, B \, X + 2 \, A^T \, X \ , \tag{B-2}$$

where N×N matrix B is real, symmetric, and positive definite, while real N×1 vector A is completely arbitrary. By completing the square, (B-2) can be written in the alternative form

$$q = (X + B^{-1} \, A)^T \, B \, (X + B^{-1} \, A) - A^T \, B^{-1} \, A \ . \tag{B-3}$$

Therefore, the minimum possible value of q is $- A^T \, B^{-1} \, A$. The case of an indefinite matrix B is undertaken below (B-42).

Random vector X is composed of correlated Gaussian random variables; its probability density function is [20; section 8-3]

$$p(X) = (2\pi)^{-N/2} \, (\det C)^{-1/2} \, \exp\left[- \frac{1}{2}(X - E)^T \, C^{-1} \, (X - E)\right] . \tag{B-4}$$

The characteristic function of interest is then given by

$$f_q(\xi) = \overline{\exp(i\xi q)} = \int dX \; p(X) \, \exp[i\xi q(X)] \ . \tag{B-5}$$

For convenience, we introduce variable

$$z = i2\xi \ . \tag{B-6}$$

Substitution of (B-2) and (B-4) in (B-5) yields

$$f_q(\xi) = (2\pi)^{-N/2} \ (\det C)^{-1/2} \times$$

$$\times \int dX \ \exp\left[-\tfrac{1}{2}(X - E)^T \ C^{-1} \ (X - E) + \tfrac{1}{2}z \ X^T \ B \ X + z \ A^T \ X\right] =$$

$$= (2\pi)^{-N/2} \ (\det C)^{-1/2} \times$$

$$\times \int dX \ \exp\left[-\tfrac{1}{2}X^T(C^{-1} - zB)X + (C^{-1} \ E + zA)^T X - \tfrac{1}{2}E^T \ C^{-1} \ E\right] \ . \tag{B-7}$$

Now we have the result [20; section 8-3]

$$\int dX \ \exp\left[-\tfrac{1}{2}X^T \ U \ X + V^T \ X\right] = (2\pi)^{N/2} \ (\det U)^{-1/2} \ \exp\left[\tfrac{1}{2}V^T \ U^{-1} \ V\right] \ . \tag{B-8}$$

This enables the reduction of the integral in (B-7) to

$$f_q(\xi) = (\det C)^{-1/2} \ \left(\det(C^{-1} - zB)\right)^{-1/2} \times$$

$$\times \exp\left[\tfrac{1}{2}(C^{-1} \ E + zA)^T(C^{-1} - zB)^{-1}(C^{-1} \ E + zA) - \tfrac{1}{2}E^T \ C^{-1} \ E\right] \ . \tag{B-9}$$

Although closed form, (B-9) is not too useful numerically because it requires an inverse of matrix $C^{-1} - zB$ for each new value of z ($= i2\xi$) of interest.

A much more compact and useful form of (B-9) can be obtained by means of the following procedure. For the given covariance matrix C and quadratic-form matrix B, solve the <u>generalized</u> characteristic-value equation

$$C\ Q = B^{-1}\ Q\ \Lambda \tag{B-10}$$

for N×N eigenvalue matrix Λ and normalized modal matrix Q, where

$$\Lambda = \mathrm{diag}[\lambda_1 \ldots \lambda_N]\ ,\quad Q = [V_1 \ldots V_N]\ ,\quad V_n = [v_{n1} \ldots v_{nN}]^T\ , \tag{B-11}$$

and N×1 column vector V_n is the n-th eigenvector with scalar components $\{v_{np}\}$. A procedure for obtaining the solutions Λ and Q to (B-10) is given in appendix E, when matrix B is positive definite.

(It should be noted that (B-10) does not have exactly the same solutions as the equation $B\ C\ Q' = Q'\ \Lambda'$; in fact, $B\ C$ is not generally symmetric, even if B is diagonal [18; page 79, (253)]. The connection is $\Lambda' = \Lambda$, $Q' = Q\ D$, with D diagonal.)

Then, we have the two very important properties [18; pages 74 - 77] of solutions Q and Λ:

$$Q^T\ B^{-1}\ Q = I\ ,\quad Q^T\ C\ Q = \Lambda\ . \tag{B-12}$$

By means of these two relations, a number of simplifications of (B-9) are possible. We begin by observing that

$$B^{-1} = Q^{-T}\ Q^{-1}\ ,\ B = Q\ Q^T\ ,\ C = Q^{-T}\ \Lambda\ Q^{-1}\ ,\ C^{-1} = Q\ \Lambda^{-1}\ Q^T. \tag{B-13}$$

(Notice that $Q^T\ Q \neq I$.) We now employ (B-13) in (B-9) to get

$$C^{-1} - zB = Q\ (\Lambda^{-1} - zI)\ Q^T\ ,\ (C^{-1} - zB)^{-1} = Q^{-T}\ (\Lambda^{-1} - zI)^{-1}\ Q^{-1}. \tag{B-14}$$

At the same time, using (B-13),

$$\det(C)\ \det(C^{-1} - z\ B) = \det(I - z\ B\ C) = \det(I - z\ Q\ \Lambda\ Q^{-1}) =$$

$$= \det(I - z\ \Lambda) = \prod_{n=1}^{N}\left(1 - z\ \lambda_n\right)\ . \qquad \text{(B-15)}$$

Meanwhile, if we denote <u>twice</u> the argument of the exponential in (B-9) by t, we have, by means of (B-14), the alternative expression

$$t = (C^{-1}\ E + zA)^T\ Q^{-T}\ (\Lambda^{-1} - zI)^{-1}\ Q^{-1}\ (C^{-1}\ E + zA) - E^T\ C^{-1}\ E\ . \qquad \text{(B-16)}$$

Now, by means of (B-13), develop the term

$$Q^{-1}\ (C^{-1}\ E + zA) = Q^{-1}\ C^{-1}\ E + zQ^{-1}\ A = \Lambda^{-1}\ Q^T\ E + zQ^T\ B^{-1}\ A =$$

$$= \Lambda^{-1}\ \underline{E} + z\underline{A}\ , \qquad \text{(B-17)}$$

where we have defined N×1 vectors

$$\underline{E} \equiv Q^T\ E = [\varepsilon_1\ \ldots\ \varepsilon_N]^T\ , \quad \varepsilon_n = V_n^T\ E\ ,$$

$$\underline{A} \equiv Q^T\ B^{-1}\ A = [\alpha_1\ \ldots\ \alpha_N]^T\ , \quad \alpha_n = V_n^T\ B^{-1}\ A\ . \qquad \text{(B-18)}$$

Then t in (B-16) becomes, again using (B-13),

$$t = (\Lambda^{-1}\ \underline{E} + z\underline{A})^T\ (\Lambda^{-1} - zI)^{-1}\ (\Lambda^{-1}\ \underline{E} + z\underline{A}) - E^T\ Q\ \Lambda^{-1}\ Q^T\ E =$$

$$= (\Lambda^{-1}\ \underline{E} + z\underline{A})^T\ (\Lambda^{-1} - zI)^{-1}\ (\Lambda^{-1}\ \underline{E} + z\underline{A}) - \underline{E}^T\ \Lambda^{-1}\ \underline{E} =$$

$$= \sum_{n=1}^{N} \frac{\left(\varepsilon_n/\lambda_n + z\ \alpha_n\right)^2}{1/\lambda_n - z} - \sum_{n=1}^{N} \frac{\varepsilon_n^2}{\lambda_n} =$$

$$= z \sum_{n=1}^{N} \frac{\left(\varepsilon_n + \alpha_n\right)^2}{1 - z\ \lambda_n} - z \sum_{n=1}^{N} \alpha_n^2\ . \qquad \text{(B-19)}$$

By collecting all these results together, we can express the characteristic function in (B-9) in the compact form

$$f_q(\xi) = \left[\prod_{n=1}^{N}\left(1-i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^{N} \frac{\left(\varepsilon_n + \alpha_n\right)^2}{1 - i2\xi\lambda_n} - i\xi \sum_{n=1}^{N} \alpha_n^2\right]. \quad \text{(B-20)}$$

The last constant in (B-20) has the alternative representation

$$-\sum_{n=1}^{N} \alpha_n^2 = -\underline{A}^T\,\underline{A} = -A^T\,B^{-1}\,Q\,Q^T\,B^{-1}\,A = -A^T\,B^{-1}\,A\,, \quad \text{(B-21)}$$

where we used (B-18) and (B-13); this is just the residual constant encountered in (B-3). Thus, the characteristic function of the leading (nonnegative) quadratic form in (B-3) is just (B-20) without the second summation inside the exponential. Additional relations available from (B-18) and (B-13) are

$$\sum_{n=1}^{N} \varepsilon_n^2 = \underline{E}^T\,\underline{E} = E^T\,Q\,Q^T\,E = E^T\,B\,E\,,$$

$$\sum_{n=1}^{N} \varepsilon_n\,\alpha_n = \underline{E}^T\,\underline{A} = E^T\,Q\,Q^T\,B^{-1}\,A = E^T\,A = A^T\,E\,. \quad \text{(B-22)}$$

However, we need the constants $\{\alpha_n\}$ and $\{\varepsilon_n\}$ individually for characteristic function (B-20). The quantities $\{2\lambda_n\}$ and $\{(\varepsilon_n + \alpha_n)^2\}$ should be computed once and stored in arrays prior to the computation of $f_q(\xi)$ in (B-20) at the numerous ξ values required.

Strictly, the leading product in (B-20) is a product of N principal-value square roots. However, it can be evaluated

numerically as a single square root of a product of the N terms $\{1-i2\xi\lambda_n\}$, provided that the location of this product is tracked in the complex plane from the point 1 when $\xi = 0$. See, for example, [21; page B-5, lines 220 - 270].

If the linear form in (B-2) is absent, then vector A = 0, $\underline{A} = 0$, and $\alpha_n = 0$ for all n. However, we still need quantities

$$\varepsilon_n = V_n^T E \quad \text{for } 1 \le n \le N , \tag{B-23}$$

for the exponent in (B-20), thereby necessitating calculation of eigenvectors $\{V_n\}$. Only when mean vector E is also zero (in addition to A) do just the eigenvalues $\{\lambda_n\}$ of (B-10) suffice for calculation of characteristic function $f_q(\xi)$ in (B-20).

In this latter case, the matrix B C can be considered instead, since it has the same eigenvalues $\{\lambda_n\}$. This follows from the following manipulations: (B-10) and (B-11) can be written as

$$C\,[V_1 \ . \ . \ . \ V_N] = B^{-1}\,[V_1 \ . \ . \ . \ V_N]\,\mathrm{diag}[\lambda_1 \ . \ . \ . \ \lambda_N] , \tag{B-24}$$

or

$$C\,V_n = B^{-1}\,V_n\,\lambda_n \quad \text{for } 1 \le n \le N . \tag{B-25}$$

Therefore

$$0 = (C - B^{-1}\lambda_n)\,V_n = B^{-1}\,(B\,C - \lambda_n I)\,V_n \tag{B-26}$$

or, since B is positive definite,

$$\det(B\,C - \lambda_n I) = 0 \quad \text{for } 1 \le n \le N . \tag{B-27}$$

Thus, $\{\lambda_n\}$ are the eigenvalues of matrix B C as well as (B-10).

CUMULANTS OF q

Before determining the cumulants of q, we present a few relations which will enable easier manipulations of various matrices encountered below. From (B-13), there follows

$$B\ C = Q\ \Lambda\ Q^{-1}\ , \quad B\ C\ B = Q\ \Lambda\ Q^{T}\ ,$$

$$B\ C\ B\ C = Q\ \Lambda^2\ Q^{-1}\ , \quad B\ C\ B\ C\ B = Q\ \Lambda^2\ Q^{T}\ . \tag{B-28}$$

Therefore

$$\mathrm{tr}(B\ C) = \mathrm{tr}(\Lambda\ Q^{-1}\ Q) = \mathrm{tr}(\Lambda) = \sum_{n=1}^{N} \lambda_n\ ,$$

$$\mathrm{tr}\left((B\ C)^2\right) = \mathrm{tr}(\Lambda^2\ Q^{-1}\ Q) = \mathrm{tr}(\Lambda^2) = \sum_{n=1}^{N} \lambda_n^2\ . \tag{B-29}$$

Now, if we expand the natural logarithm of the general characteristic function $f_q(\xi)$ in (B-20) in a power series in $i\xi$, we can easily pick off the cumulants as

$$\chi_q(1) = \mu_q = \sum_{n=1}^{N}\left(\lambda_n + \varepsilon_n^2 + 2\varepsilon_n\alpha_n\right) = \mathrm{tr}(B\ C) + E^T\ B\ E + 2\ A^T\ E\ ,$$

$$\chi_q(p) = 2^{p-1}\ (p-1)!\ \sum_{n=1}^{N} \lambda_n^{p-1}\left(\lambda_n + p(\varepsilon_n + \alpha_n)^2\right) \text{ for } p \geq 2. \tag{B-30}$$

Here, we used (B-29) and (B-22). In particular, the variance of q is

$$\chi_q(2) = \sigma_q^2 = 2\sum_{n=1}^{N}\left(\lambda_n^2 + 2\lambda_n(\varepsilon_n + \alpha_n)^2\right) =$$

$$= 2\ \mathrm{tr}\left((B\ C)^2\right) + 4\ (B\ E + A)^T\ C\ (B\ E + A)\ . \tag{B-31}$$

This latter relation follows from these manipulations:

$$\sum_{n=1}^{N} \lambda_n(\varepsilon_n + \alpha_n)^2 = (\underline{E} + \underline{A})^T \Lambda (\underline{E} + \underline{A}) =$$

$$= (E + B^{-1}A)^T Q \Lambda Q^T (E + B^{-1}A) = (E + B^{-1}A)^T B C B (E + B^{-1}A) =$$

$$= (B E + A)^T C (B E + A) . \tag{B-32}$$

Here, we used (B-18) and (B-28).

ALTERNATIVE DERIVATION OF CHARACTERISTIC FUNCTION (B-20)

We start again with conditions (B-1) and (B-2). Then, solve (B-10) for Q and Λ, as before. Now, consider the linear transformation of Gaussian random vector X according to

$$Y = Q^T (X - E) = [y_1 \; . \; . \; . \; y_N]^T . \tag{B-33}$$

Then N×1 Gaussian vector Y has mean $\overline{Y} = 0$ and covariance matrix

$$\overline{Y\, Y^T} = Q^T \, \overline{(X - E)(X - E)^T} \, Q = Q^T C Q = \Lambda , \tag{B-34}$$

where we used (B-1) and (B-12). This diagonal matrix means that

$$\overline{y_n} = 0 , \quad \overline{y_n \, y_m} = \lambda_n \, \delta_{nm} ; \tag{B-35}$$

that is, $\{y_n\}$ are zero-mean uncorrelated (and therefore <u>independent</u>) Gaussian random variables. This is the key to this development.

At the same time, solving (B-33) for X, we have, with (B-12),

$$X = E + Q^{-T}\, Y = E + B^{-1}\, Q\, Y\ . \tag{B-36}$$

Then quadratic form (B-2) can be expressed as

$$\begin{aligned} q &= (E + B^{-1}\, Q\, Y)^T\, B\, (E + B^{-1}\, Q\, Y) + 2\, A^T\, (E + B^{-1}\, Q\, Y) = \\ &= Y^T\, Y + 2\, (E^T\, Q + A^T\, B^{-1}\, Q)\, Y + E^T\, B\, E + 2\, A^T\, E = \\ &= Y^T\, Y + 2\, (\underline{E}^T + \underline{A}^T)\, Y + E^T\, B\, E + 2\, A^T\, E = \\ &= \sum_{n=1}^{N} \left(y_n^2 + 2(\varepsilon_n + \alpha_n) y_n + \varepsilon_n^2 + 2\varepsilon_n \alpha_n \right)\ , \end{aligned} \tag{B-37}$$

where we used (B-12), (B-18), and (B-22). Therefore, using the independence property derived in (B-34) and (B-35), the characteristic function of q is

$$\begin{aligned} f_q(\xi) &= \overline{\exp(i\xi q)} = \prod_{n=1}^{N} \left\{ \int \frac{dy_n}{(2\pi\lambda_n)^{1/2}} \exp\left(- \frac{y_n^2}{2\lambda_n} \right) \times \right. \\ &\quad \left. \times \exp\left[i\xi \left(y_n^2 + 2(\varepsilon_n + \alpha_n) y_n + \varepsilon_n^2 + 2\varepsilon_n \alpha_n \right) \right] \right\} = \\ &= \prod_{n=1}^{N} \left\{ \left(1 - i2\xi\lambda_n \right)^{-1/2} \exp\left[i\xi \frac{(\varepsilon_n + \alpha_n)^2}{1 - i2\xi\lambda_n} - i\xi\alpha_n^2 \right] \right\}\ , \end{aligned} \tag{B-38}$$

which is equal to (B-20).

PROPERTIES OF EIGENVECTORS $\{V_n\}$

Consider the representation of modal matrix Q in terms of eigenvectors $\{V_n\}$ in (B-11). Then the first relation in (B-12) and the second relation in (B-13) yield, respectively,

$$V_n^T\ B^{-1}\ V_m = \delta_{nm}\ , \qquad \sum_{n=1}^{N} V_n\ V_n^T = B\ . \tag{B-39}$$

Similarly, the second relation in (B-12) and the fourth relation in (B-13) yield, respectively,

$$V_n^T\ C\ V_m = \lambda_n\ \delta_{nm}\ , \qquad \sum_{n=1}^{N} \frac{1}{\lambda_n}\ V_n\ V_n^T = C^{-1}\ . \tag{B-40}$$

Thus, there are two orthogonality relations satisfied by the eigenvectors $\{V_n\}$, namely the leading relations in (B-39) and (B-40). If, in addition, matrix B or C is diagonal, then the trailing relations in (B-39) and (B-40) yield an additional orthogonality property.

In terms of the components (B-11), the relations above become

$$\sum_{k,j=1}^{N} v_{nk}\ \left(B^{-1}\right)_{kj}\ v_{mj} = \delta_{nm}\ , \qquad \sum_{n=1}^{N} v_{nk}\ v_{nj} = \left(B\right)_{kj}\ , \tag{B-41}$$

and

$$\sum_{k,j=1}^{N} v_{nk}\ \left(C\right)_{kj}\ v_{mj} = \lambda_n\ \delta_{nm}\ , \quad \sum_{n=1}^{N} \frac{1}{\lambda_n}\ v_{nk}\ v_{nj} = \left(C^{-1}\right)_{kj}\ . \tag{B-42}$$

GENERAL SYMMETRIC MATRIX B

All the results above are based on the premise that matrix B in the quadratic form in (B-2) is positive definite. However, there are examples where this is not the situation, in which case the corresponding procedure for solution of (B-10) in appendix E (that was mentioned under (B-11)) is not applicable. We now give an alternative procedure for obtaining the characteristic function of q in (B-2) for <u>any</u> real matrix B, whether definite or not. Matrix B can be taken symmetric without loss of generality, since only the symmetric part of B is active in (B-2).

We first observe that the N×N covariance matrix C in (B-1) is always nonnegative definite because

$$V^T \, C \, V = V^T \, \overline{(X - E)(X - E)^T} \, V = \overline{\left(V^T (X - E)\right)^2} \geq 0 \tag{B-43}$$

for any N×1 real vector V. We shall presume that C is positive definite. Instead of solving (B-10), we solve the alternative generalized real characteristic-value matrix equation

$$B \, Q = C^{-1} \, Q \, \Lambda \tag{B-44}$$

for N×N normalized modal matrix Q and diagonal eigenvalue matrix Λ. A procedure for this solution is given in appendix E; see (E-14) and sequel.

Then, these solutions satisfy [18; pages 74 - 77]

$$Q^T \, C^{-1} \, Q = I \; , \tag{B-45}$$

$$Q^T \, B \, Q = \Lambda \equiv \mathrm{diag}[\lambda_1 \; . \; . \; . \; \lambda_N] \; . \tag{B-46}$$

Now let linearly transformed N×1 random vector

$$Z = Q^{-1} (X - E) \equiv [z_1 \; . \; . \; . \; z_N]^T \; . \tag{B-47}$$

Then mean $\overline{Z} = 0$ and covariance matrix

$$\overline{Z \; Z^T} = Q^{-1} \; \overline{(X - E)(X - E)^T} \; Q^{-T} = Q^{-1} \; C \; Q^{-T} = I \; , \tag{B-48}$$

upon use of (B-1) and (B-45). Solve (B-47) for X, obtaining

$$X = Q \; Z + E \; . \tag{B-49}$$

Now, substitute (B-49) into quadratic and linear form (B-2):

$$\begin{aligned} q &= (Q \; Z + E)^T \; B \; (Q \; Z + E) + 2 \; A^T \; (Q \; Z + E) = \\ &= (Z + Q^{-1} \; E)^T \; Q^T \; B \; Q \; (Z + Q^{-1} \; E) + 2 \; A^T \; Q \; (Z + Q^{-1} \; E) = \\ &= (Z + \underline{E})^T \; \Lambda \; (Z + \underline{E}) + 2 \; \underline{A}^T \; (Z + \underline{E}) \; , \end{aligned} \tag{B-50}$$

where we used (B-46) and defined two auxiliary N×1 vectors

$$\underline{E} = Q^{-1} \; E \equiv [\varepsilon_1 \; . \; . \; . \; \varepsilon_N]^T \tag{B-51}$$

and

$$\underline{A} = Q^T \; A \equiv [\alpha_1 \; . \; . \; . \; \alpha_N]^T \; . \tag{B-52}$$

Therefore, (B-50) can now be expressed as

$$q = \sum_{n=1}^{N} (\lambda_n \; z_n + \lambda_n \; \varepsilon_n + 2 \; \alpha_n)(z_n + \varepsilon_n) \; . \tag{B-53}$$

The characteristic function of q can now be readily found, with the assistance of covariance property (B-48), in its most compact form

$$f_q(\xi) = \overline{\exp(i\xi q)} =$$

$$= \prod_{n=1}^{N} \left(\int dz_n \; (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2} z_n^2 + i\xi \left(\lambda_n z_n + \lambda_n \varepsilon_n + 2\alpha_n\right)\left(z_n + \varepsilon_n\right)\right]\right) =$$

$$= \left[\prod_{n=1}^{N} \left(1 - i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^{N} \frac{\lambda_n \varepsilon_n^2 + 2\varepsilon_n \alpha_n + i2\xi\alpha_n^2}{1 - i2\xi\lambda_n}\right] . \qquad \text{(B-54)}$$

In summary, the following computations must be performed: solve (B-44) for N×N matrices Q and Λ; compute N×1 vectors $\underline{E}$ and $\underline{A}$ by means of (B-51) and (B-52), respectively; and evaluate $f_q(\xi)$ at desired ξ values by use of (B-54). The only inverse matrix required is that of modal matrix Q; the solution of (B-44) does not actually require calculation of C^{-1}, as will be seen in appendix E, (E-14) and sequel.

If mean vector E in (B-1) is zero and/or if linear form vector A in (B-2) is zero, the corresponding calculations in (B-51) and (B-52) can be circumvented. If both E and A are zero, the exp term in (B-54) is absent altogether.

In this latter case, only the eigenvalues $\{\lambda_n\}$ of (B-44) need be determined. If we let V_n be the n-th eigenvector (column) of Q, equation (B-44) takes the form

$$B\, V_n = C^{-1}\, V_n\, \lambda_n \quad \text{for } 1 \le n \le N . \qquad \text{(B-55)}$$

This can be manipulated into

$$(B - \lambda_n\, C^{-1}) V_n = 0 , \qquad \text{(B-56)}$$

meaning that

$$\det(B - \lambda_n C^{-1}) = 0 \tag{B-57}$$

or, since $\det(C) \neq 0$, that

$$\det(B\ C - \lambda_n I) = 0 \ . \tag{B-58}$$

That is, $\{\lambda_n\}$ are the eigenvalues of matrix B C. This is a much simpler numerical task than solving (B-44) for both Q and Λ.

If none of the eigenvalues $\{\lambda_n\}$ are zero or near zero, then (B-46) furnishes inverse matrix $Q^{-1} = \Lambda^{-1}\ Q^T\ B$ and an alternative to (B-51), namely

$$\underline{E} = \Lambda^{-1}\ Q^T\ B\ E \ ,$$

$$\varepsilon_n = \frac{1}{\lambda_n}\ V_n^T\ B\ E \quad \text{for } 1 \le n \le N \ . \tag{B-59}$$

Also, from (B-52), regardless of the sizes of the eigenvalues,

$$\alpha_n = V_n^T\ A \quad \text{for } 1 \le n \le N \ . \tag{B-60}$$

The interrelationships between Q and Λ in (B-44) and the corresponding matrices Q and Λ in (B-10) are derived in appendix E, (E-24) and sequel. The identity of characteristic functions (B-20) and (B-54) is also verified there.

PROPERTIES OF EIGENVECTORS $\{V_n\}$

Relations (B-45) and (B-46) yield the following orthogonality properties between the eigenvectors:

$$V_n^T \, C^{-1} \, V_m = \delta_{nm} \,, \qquad V_n^T \, B \, V_m = \lambda_n \, \delta_{nm} \,. \qquad \text{(B-61)}$$

Also, since (B-45) and (B-46) can be alternatively expressed as $C = Q \, Q^T$ and $B^{-1} = Q \, \Lambda^{-1} \, Q^T$, we have

$$\sum_{n=1}^{N} V_n \, V_n^T = C \,, \qquad \sum_{n=1}^{N} \frac{1}{\lambda_n} \, V_n \, V_n^T = B^{-1} \,. \qquad \text{(B-62)}$$

As a special case of (B-61), it can be seen that $\lambda_n = V_n^T \, B \, V_n$; therefore, if matrix B is nonnegative definite, then $\lambda_n \geq 0$ for all n. The following example demonstrates that indefinite symmetric B matrices can lead to negative eigenvalues, even when matrix C is symmetric and positive definite:

$$B = \begin{bmatrix} 4 & -6 \\ -6 & 7 \end{bmatrix} , \qquad C = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} ; \qquad \text{(B-63)}$$

the eigenvalue and eigenvector solutions to (B-44) are

$$\Lambda = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} , \qquad Q = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} . \qquad \text{(B-64)}$$

CUMULANTS OF q

By taking the logarithm of characteristic function $f_q(\xi)$ in (B-54) and expanding in a power series in $i\xi$, the cumulants of quadratic and linear form q are found to be

$$\chi_q(m) = \left\{ \begin{array}{ll} \sum_{n=1}^{N} \left[\lambda_n\left(1 + \varepsilon_n^2\right) + 2\varepsilon_n\alpha_n\right] & \text{for } m = 1 \\ (m-1)!\ 2^{m-1} \sum_{n=1}^{N} \lambda_n^{m-2}\left[\lambda_n^2 + m\left(\lambda_n\varepsilon_n+\alpha_n\right)^2\right] & \text{for } m \geq 2 \end{array} \right\}. \quad \text{(B-65)}$$

APPENDIX C. CHARACTERISTIC FUNCTION OF OUTPUT γ FOR GENERAL TRANSMITTED SIGNAL ENERGIES $\{\tilde{E}_k\}$ AND RECEIVER WEIGHTS $\{A_k\}$

The conditional characteristic function of processor output γ, for arbitrary transmitted signal energies $\{\tilde{E}_k\}$ and receiver weights $\{A_k\}$, was given in (27) as

$$f_c(\xi) = \prod_{k=1}^{K} \left(1 - i\xi\, 2\sigma_k^2\right)^{-1} \exp\left[i\xi \sum_{k=1}^{K} w_k\, r_k^2\right] , \qquad \text{(C-1)}$$

where

$$\sigma_k^2 = 2\, N_o\, A_k^2\, \tilde{E}_k , \qquad w_k = \frac{4\, A_k^2\, \tilde{E}_k^2}{1 - i\xi\, 4\, N_o\, A_k^2\, \tilde{E}_k} . \qquad \text{(C-2)}$$

It is important to notice that the coefficients $\{w_k\}$ in the summation in (C-1) are complex and are functions of argument ξ.

Now, we want to find the <u>unconditional</u> characteristic function of γ, after averaging over the statistics of amplitude scalings $\{r_k\}$. From (43) - (45),

$$r_k^2 = \sum_{m=1}^{M} \left[c_m(t_k,f_k) + g_m(t_k,f_k)\right]^2 = \sum_{m=1}^{M} x_{mk}^2 , \qquad \text{(C-3)}$$

where Gaussian random variables

$$x_{mk} = c_m(t_k,f_k) + g_m(t_k,f_k) \quad \text{for } 1 \le m \le M , \quad 1 \le k \le K . \qquad \text{(C-4)}$$

Therefore, the exponential term in (C-1) becomes

$$\exp\left[i\xi \sum_{k=1}^{K} w_k\, r_k^2\right] = \exp\left[i\xi\, X^T\, \tilde{W}\, X\right] , \qquad \text{(C-5)}$$

where MK×1 random Gaussian real column vector

$$X = [x_{11} \cdots x_{M1} \ \cdots \ x_{1k} \cdots x_{Mk} \ \cdots \ x_{1K} \cdots x_{MK}]^T , \tag{C-6}$$

and MK×MK diagonal matrix of complex elements

$$\tilde{W} = \text{diag}[w_1 \cdots \cdot w_1 \ \cdots \ \underbrace{w_k \cdots \cdot w_k}_{\text{M terms}} \ \cdots \ w_K \cdots \cdot w_K] . \tag{C-7}$$

The MK×MK covariance matrix of vector X is

$$K_x = \text{Cov}\{X\} = \overline{(X - \overline{X}) \ (X - \overline{X})^T} \tag{C-8}$$

which is known.

The unconditional characteristic function of γ is available from (C-1) and (C-5) in the form

$$f_\gamma(\xi) = \overline{f_c(\xi)} = \prod_{k=1}^{K} \left(1 - i\xi \ 2\sigma_k^2\right)^{-1} \overline{\exp\left[i\xi \ X^T \ \tilde{W} \ X\right]} =$$

$$= \prod_{k=1}^{K} \left(1 - i\xi \ 2\sigma_k^2\right)^{-1} \int dX \ (2\pi)^{-MK/2} \left(\det \ K_x\right)^{-1/2} \times$$

$$\times \exp\left[- \frac{1}{2}(X - \overline{X})^T \ K_x^{-1} \ (X - \overline{X}) + i\xi \ X^T \ \tilde{W} \ X\right] =$$

$$= \prod_{k=1}^{K} \left(1 - i\xi \ 2\sigma_k^2\right)^{-1} \left(\det\left(I - i\xi 2 \ \tilde{W} \ K_x\right)\right)^{-1/2} \times$$

$$\times \exp\left[\frac{1}{2} \ \overline{X}^T \ (I - i\xi 2 \ \tilde{W} \ K_x)^{-1} \ K_x^{-1} \ \overline{X} - \frac{1}{2} \ \overline{X}^T \ K_x^{-1} \ \overline{X}\right] . \tag{C-9}$$

The matrix $\tilde{W}$ depends on ξ through coefficients $\{w_k\}$ in (C-2). Therefore, in order to compute (C-9), it is necessary to invert a complex MK×MK matrix for each ξ of interest. However, if mean $\overline{X}$ is zero, only the determinant in (C-9) need be evaluated.

UNCORRELATED FADING COMPONENTS $\{g_m(t,f)\}$

In this subsection, we presume that the fading components in model (45) are uncorrelated and therefore independent. This means that, in (C-4), random variables $x_{m1},\ldots,x_{mK}$ are independent of $x_{n1},\ldots,x_{nK}$ for $m \neq n$. Reference to (C-1) and (C-3) then yields the characteristic function of output γ in the form

$$f_\gamma(\xi) = \overline{f_c(\xi)} = \prod_{k=1}^{K}\left(1 - i\xi\, 2\sigma_k^2\right)^{-1} E(\xi) \; , \qquad \text{(C-10)}$$

where

$$E(\xi) \equiv \overline{\exp\left[i\xi \sum_{k=1}^{K} w_k\, r_k^2\right]} = \overline{\exp\left[i\xi \sum_{k=1}^{K} w_k \sum_{m=1}^{M} x_{mk}^2\right]} =$$

$$= \prod_{m=1}^{M} \overline{\exp\left[i\xi \sum_{k=1}^{K} w_k\, x_{mk}^2\right]} \; . \qquad \text{(C-11)}$$

Now, let K×1 real random vector X_m and K×K complex diagonal matrix W be defined according to

$$X_m = [x_{m1} \cdot \cdot \cdot x_{mK}]^T \; , \qquad W = \mathrm{diag}[w_1 \cdot \cdot \cdot w_K] \; . \qquad \text{(C-12)}$$

Also, let the covariance matrix of X_m be denoted by K_m, which is a K×K real matrix, for $1 \le m \le M$. Then, the m-th term in (C-11) becomes

$$\overline{\exp\left[i\xi \sum_{k=1}^{K} w_k\, x_{mk}^2\right]} = \overline{\exp\left[i\xi\, X_m^T W X_m\right]} = \int dX_m\, (2\pi)^{-K/2} \times$$

$$\times \, (\det K_m)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(X_m - \overline{X}_m\right)^T K_m^{-1}\left(X_m - \overline{X}_m\right) + i\xi\, X_m^T W X_m\right] =$$

$$\left(\det\left(I-i\xi 2WK_m\right)\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\,\overline{X}_m^T\left(I - i\xi 2\;W\;K_m\right)^{-1} K_m^{-1}\;\overline{X}_m - \frac{1}{2}\,\overline{X}_m^T\,K_m^{-1}\,\overline{X}_m\right] \tag{C-13}$$

Finally, the desired unconditional characteristic function of processor output γ follows from a combination of (C-10), (C-11), and (C-13) as

$$f_\gamma(\xi) = \prod_{k=1}^{K}\left(1 - i\xi\;2\sigma_k^2\right)^{-1} \prod_{m=1}^{M}\left\{\left(\det\left(I - i\xi 2\;W\;K_m\right)\right)^{-\frac{1}{2}} \times\right.$$

$$\left. \times \exp\left[\frac{1}{2}\,\overline{X}_m^T\left(I - i\xi 2\;W\;K_m\right)^{-1} K_m^{-1}\;\overline{X}_m - \frac{1}{2}\,\overline{X}_m^T\,K_m^{-1}\,\overline{X}_m\right]\right\} . \tag{C-14}$$

The major computational burden here consists of the inverses of M complex K×K matrices for each ξ value of interest. However, if the M covariance matrices $\{K_m\}$ are all identical, the task is considerably simplified. Alternatively, if all the means $\{\overline{X}_m\}$ are zero, then the evaluation is limited to M determinants of the complex M×M matrices $I - i\xi 2\;W\;K_m$. The K×K complex matrix W is defined by (C-12) and (C-2).

APPENDIX D. DERIVATION OF PROBABILITY DENSITY AND EXCEEDANCE DISTRIBUTION FROM CHARACTERISTIC FUNCTION

The general forms in (136) and (158) of the characteristic function of processor output γ can be written as

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + T_{mk}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{e_{mk}}{1 - i2\xi\left(1 + T_{mk}\right)}\right], \qquad \text{(D-1)}$$

where, respectively

$$T_{mk} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)} \quad \text{or} \quad T_{mk} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k \,, \qquad e_{mk} = \underline{\varepsilon}_k^{(m)2} . \qquad \text{(D-2)}$$

We want tc determine the corresponding probability density function and exceedance distribution function. There is a possible numerical problem using Fourier transforms on (D-1) directly, since its asymptotic decay is only proportional to ξ^{-K} as $\xi \to \infty$, which makes accurate results difficult to achieve for small K.

We will expand (D-1) in a power series in $(1 - i2\xi)^{-1}$ and thereby obtain not only an asymptotic expansion of $f_\gamma(\xi)$, but in fact, a convergent series which can be transformed term-by-term to yield the probability density function of γ. Integration then yields the exceedance distribution function. Furthermore, it will turn out that all the series coefficients are positive and related by recursions involving only positive terms. These properties enable very accurate and efficient evaluation.

We begin by making the transformation

$$z = \frac{1}{1 - i2\xi} , \quad i2\xi = - \frac{1 - z}{z} . \tag{D-3}$$

Substitution in (D-1) yields, after manipulation,

$$f_\gamma(\xi) = F\, z^K\, D(z)\, E(z) , \tag{D-4}$$

where

$$F = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 + T_{mk}\right)\right]^{-\frac{1}{2}} , \quad D(z) = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - z\, r_{mk}\right)\right]^{-\frac{1}{2}} ,$$

$$r_{mk} = \frac{T_{mk}}{1 + T_m} , \quad E(z) = \exp\left[- \frac{1 - z}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}/(1+T_{mk})}{1 - z\, r_{mk}}\right] . \tag{D-5}$$

To develop the series of D(z) in powers of z, expand

$$\ln D(z) = - \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \ln(1 - z\, r_{mk}) = \sum_{p=1}^{\infty} \beta_p\, z^p , \tag{D-6}$$

where

$$\beta_p = \frac{1}{2p} \sum_{m=1}^{M} \sum_{k=1}^{K} r_{mk}^p \quad \text{for } p \geq 1 . \tag{D-7}$$

Notice that $\beta_p \geq 0$ for $p \geq 1$ since $T_{mk} \geq 0$; this latter property follows from (D-2) and the fact that eigenvalues of covariance matrices can never be negative.

At the same time, the expansion of ln E(z) in (D-5) is

$$\ln E(z) = - \frac{1}{2}(1 - z) \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 + T_{mk}} \sum_{p=0}^{\infty} r_{mk}^p\, z^p . \tag{D-8}$$

By combining the results in (D-6) - (D-8), there follows, after some manipulations,

$$\ln(D(z)\ E(z)) = \sum_{p=0}^{\infty} \alpha_p\ z^p\ , \tag{D-9}$$

where

$$\alpha_o = -\frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 + T_{mk}}\ ,$$

$$\alpha_p = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{T_{mk}^{p-1}}{(1 + T_{mk})^p} \left[\frac{T_{mk}}{p} + \frac{e_{mk}}{1 + T_{mk}}\right] \quad \text{for } p \geq 1\ . \tag{D-10}$$

Notice that $\alpha_p \geq 0$ for $p \geq 1$.

The desired product for use in (D-4) is then [17; page 93]

$$D(z)\ E(z) = \exp\left(\sum_{p=0}^{\infty} \alpha_p\ z^p\right) = \sum_{p=0}^{\infty} g_p\ z^p\ , \tag{D-11}$$

where

$$g_o = \exp(\alpha_o)\ , \qquad g_p = \frac{1}{p} \sum_{n=1}^{p} n\ \alpha_n\ g_{p-n} \quad \text{for } p \geq 1\ . \tag{D-12}$$

It is important to notice that $g_p \geq 0$ for $p \geq 0$; that is, the recursion in (D-12) involves no negative quantities, thereby avoiding cancellation error.

Finally, the characteristic function in (D-4) can be expressed in the desired power series in $z\ (= (1 - i2\xi)^{-1})$ as

$$f_\gamma(\xi) = F\ z^K \sum_{p=0}^{\infty} g_p\ z^p\ . \tag{D-13}$$

Now, we know the following Fourier transform pair between a characteristic function and a probability density function:

$$z^n = \frac{1}{(1 - i2\xi)^n} \longleftrightarrow \frac{u^{n-1} \exp(-u/2)}{2^n \, (n-1)!} \quad \text{for } u > 0 \; . \qquad \text{(D-14)}$$

This enables us to write the probability density function of γ, directly from (D-13), in the form

$$p_\gamma(u) = F \sum_{p=0}^{\infty} g_p \frac{u^{K+p-1} \exp(-u/2)}{2^{K+p} \, (K+p-1)!} \quad \text{for } u > 0 \; , \qquad \text{(D-15)}$$

where scale factor F is given by (D-5). All the terms in this series are nonnegative.

Integration on the positive tail of density p_γ gives the exceedance distribution function of γ:

$$\text{Prob}(\gamma > u) = \int_u^\infty dt \; p_\gamma(t) = F \sum_{p=0}^{\infty} g_p \; H_{K+p-1}\left(\frac{u}{2}\right) \quad \text{for } u > 0 \; , \; \text{(D-16)}$$

where F is given by (D-5) and

$$H_n(x) \equiv \exp(-x) \sum_{k=0}^{n} \frac{x^k}{k!} \quad \text{for } n \geq 0 \; , \; x \geq 0 \; . \qquad \text{(D-17)}$$

This latter sequence of functions is easily generated by use of the coupled recurrences

$$H_o(x) = \exp(-x) \; , \quad H_n(x) = T_n(x) + H_{n-1}(x) \quad \text{for } n \geq 1 \; ,$$

$$T_o(x) = \exp(-x) \; , \quad T_n(x) = T_{n-1}(x) \, \frac{x}{n} \quad \text{for } n \geq 1 \; . \qquad \text{(D-18)}$$

The two recurrences derived above, namely (D-12) and (D-18), utilize only nonnegative quantities. The single negative term, α_o in (D-10), is immediately converted to positive quantity g_o in (D-12) and α_o is not encountered again.

Since processor output γ can never be negative (see figure 1), we have, from (D-16) and (D-17),

$$1 = \text{Prob}(\gamma > 0) = F \sum_{p=0}^{\infty} g_p \ . \tag{D-19}$$

This relation can be used to furnish an upper bound on the error incurred by using up to the p = N term in series (D-16) for the exceedance distribution function. In particular, the error is

$$F \sum_{p=N+1}^{\infty} g_p \; H_{K+p-1}\left(\frac{u}{2}\right) \le F \sum_{p=N+1}^{\infty} g_p = 1 - F \sum_{p=0}^{N} g_p \tag{D-20}$$

for all $u \ge 0$. The upper bound of 1 on $H_n(x)$ follows immediately from definition (D-17). Since scale factor F and coefficients $\{g_p\}$ for $0 \le p \le N$ must be computed anyway, in order to utilize (D-16), error bound (D-20) is simple to compute. It says that the error at any u is never larger than the error at the origin.

However, (D-16) converges too slowly in some cases of large signal-to-noise ratio; only sequence $\{g_p\}$ converges to zero. Result (D-19) allows a modification to sum (D-16) which will converge much more rapidly. From (D-17), since $H_n(x) \to 1$ as $n \to \infty$, we express (D-16) as

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p \left[H_{K+p-1}\left(\frac{u}{2}\right) - 1 + 1 \right] =$$

$$= 1 - F \sum_{p=0}^{\infty} g_p \left[1 - H_{K+p-1}\left(\frac{u}{2}\right) \right] . \qquad \text{(D-21)}$$

Now, <u>both</u> factors in the latter sum decay to zero as p increases. In fact,

$$1 - H_{n-1}(x) \le \exp(-x) \frac{x^n}{n!} \frac{n + 1}{n + 1 - x} \quad \text{for } n > x . \qquad \text{(D-22)}$$

This alternative in (D-21) is utilized in the program listed at the end of this appendix. Recursions for $1 - H_n(x)$, which are obvious modifications of (D-18), are also used.

As noted at the beginning of this appendix, form (D-1) encompasses (136) and (158) in the main text. It therefore also covers (139) and (160), because these are reductions that can be obtained, respectively, by setting all $\{e_{mk}\}$ to zero. If all the $\{e_{mk}\}$ <u>and</u> $\{T_{mk}\}$ are zero, then all $\{\alpha_p\}$ are zero, from (D-10). Then, all $\{g_p\}$ are zero except $g_o = 1$, from (D-12). In this case of noise-only, series (D-16) has just the one term $H_{K-1}(u/2)$.

The results above, for the expansions of the characteristic function, probability density function, and exceedance distribution function, also apply directly to forms (124) (and (126)) as well as (144) (and (149)) in the main text, but with the following changes. For (124), define instead

$$T_n = \frac{2\tilde{E}}{N_o} \lambda_n , \quad e_n = \frac{2\tilde{E}}{N_o} \varepsilon_n^2 , \quad F = \left[\prod_{n=1}^{KM} \left(1 + T_n\right) \right]^{-\frac{1}{2}} , \qquad \text{(D-23)}$$

along with

$$\alpha_o = -\frac{1}{2}\sum_{n=1}^{KM} \frac{e_n}{1 + T_n} ,$$

$$\alpha_p = \frac{1}{2}\sum_{n=1}^{KM} \frac{T_n^{p-1}}{(1 + T_n)^p} \left[\frac{T_n}{p} + \frac{e_n}{1 + T_n}\right] \quad \text{for } p \geq 1 . \tag{D-24}$$

The recurrence (D-12) involving $\{g_p\}$ is unchanged.

For (144), define instead

$$T_k = \frac{2E_{11}}{N_o} \underline{\lambda}_k , \qquad F = \left[\prod_{k=1}^{K}\left(1 + T_k\right)\right]^{-M/2} , \tag{D-25}$$

along with

$$\alpha_o = -\frac{1}{2}\sum_{k=1}^{K} \frac{\underline{h}_k}{1 + T_k} ,$$

$$\alpha_p = \frac{1}{2}\sum_{k=1}^{K} \frac{T_k^{p-1}}{(1 + T_k)^p} \left[\frac{M}{p}\, T_k + \frac{\underline{h}_k}{1 + T_k}\right] \quad \text{for } p \geq 1 . \tag{D-26}$$

Again, the recurrence involving $\{g_p\}$ is unchanged.

REFINEMENT

In some cases, a refinement of the above procedure may be worthwhile, especially for larger signal-to-noise ratios. We illustrate it with reference to (150), which applies to M = 2, two components in fading model (45). In this case, we have characteristic function

$$f_\gamma(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi Q_k\right)\right]^{-1} , \tag{D-27}$$

where

$$Q_k = 1 + \frac{2E_{11}}{N_o} \underline{\lambda}_k \quad \text{for } 1 \le k \le K . \tag{D-28}$$

These quantities $\{Q_k\}$ are always larger than 1 for nonzero signal-to-noise ratio.

Instead of making transformation (D-3), we modify it to the more general form

$$z = \frac{1}{1 - i2\xi Q} , \tag{D-29}$$

where Q can be chosen larger than 1 if desired. Then, defining

$$F = Q^K \left[\prod_{k=1}^{K} Q_k\right]^{-1} , \quad t_k = \frac{Q_k - Q}{Q_k} \quad \text{for } 1 \le k \le K , \tag{D-30}$$

characteristic function (D-27) can be developed as

$$f_\gamma(\xi) = F\, z^K \left[\prod_{k=1}^{K}\left(1 - t_k z\right)\right]^{-1} = F\, z^K \exp\left(\sum_{p=1}^{\infty} \alpha_p\, z^p\right) , \tag{D-31}$$

now with

$$p\ \alpha_p = \sum_{k=1}^{K} t_k^p = \sum_{k=1}^{K} \left(\frac{Q_k - Q}{Q_k}\right)^p \quad \text{for } p \geq 1 \ . \tag{D-32}$$

If we take the constant Q to have value

$$\tilde{Q} = \min_{1 \leq k \leq K} Q_k \ , \tag{D-33}$$

then all the $\{t_k\}$ and $\{\alpha_p\}$ are nonnegative, and we still have all the desirable properties listed above. The value $\tilde{Q}$ is always larger than 1; it enables (D-31) to furnish a better fit to (D-27) with fewer terms than arbitrarily forcing $Q = 1$. Larger values than $\tilde{Q}$ can yield still <u>more</u>-rapidly convergent series, but at the expense of some negative $\{g_p\}$. There follows, from (D-31)

$$f_\gamma(\xi) = F\ z^K \sum_{p=0}^{\infty} g_p\ z^p = F \sum_{p=0}^{\infty} g_p\ (1 - i2\xi Q)^{-K-p} \tag{D-34}$$

for any Q; see (D-11) and (D-12) for the $\{g_p\}$. Expansion (D-34) should be compared with the moment procedure in [17; (102) and (105)], where a different basis set, namely the generalized Laguerre functions, is used for the series expansion.

The corresponding exceedance distribution function is

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p\ H_{K+p-1}\left(\frac{u}{2Q}\right) \quad \text{for } u > 0 \ . \tag{D-35}$$

Alternatively, we have the more rapidly convergent form

$$\text{Prob}(\gamma > u) = 1 - F \sum_{p=0}^{\infty} g_p \left[1 - H_{K+p-1}\left(\frac{u}{2Q}\right)\right] \quad \text{for } u > 0 \ . \tag{D-36}$$

PROBABILITY DENSITY FROM CHARACTERISTIC FUNCTION (163)

The characteristic function of normalized random variable $\phi = q/R_{11}(0,0)$ defined in (161) was presented in (163). The corresponding probability density function $p_\phi(u)$ was then evaluated for a few special cases in (166), (171), (175), and (180). More generally, if we apply the expansion technique above to general result (163), we obtain the following procedure for $p_\phi(u)$: given M, $\{r^{(m)}\}$, $\{h_m^2\}$, and threshold u,

$$F = \left(\prod_{m=1}^{M} r^{(m)}\right)^{-\frac{1}{2}} , \qquad \alpha_o = -\frac{1}{2}\sum_{m=1}^{M} h_m^2 , \tag{D-37}$$

$$p\ \alpha_p = \frac{1}{2}\sum_{m=1}^{M} \frac{\left(r^{(m)} - 1\right)^{p-1}}{r^{(m)p}} \left(r^{(m)} - 1 + p\ h_m^2\right) \quad \text{for } p \geq 1 , \tag{D-38}$$

$$g_o = \exp(\alpha_o) , \qquad g_p = \frac{1}{p}\sum_{n=1}^{p} n\ \alpha_n\ g_{p-n} \quad \text{for } p \geq 1 , \tag{D-39}$$

$$h_o(u) = \frac{u^{\frac{1}{2}M-1} \exp(-u/2)}{2^{M/2}\ \Gamma(M/2)} ,$$

$$h_p(u) = h_{p-1}(u)\ \frac{u}{M - 2 + 2p} \quad \text{for } p \geq 1 , \tag{D-40}$$

$$p_\phi(u) = F \sum_{p=0}^{\infty} g_p\ h_p(u) \quad \text{for } u > 0 . \tag{D-41}$$

A function subprogram for this particular procedure is listed on the next page.

FUNCTION SUBPROGRAM FOR (D-37) - (D-41)

```
10    DEF FNPdf(DOUBLE M,REAL U,Rs(*),Etasq(*))
20    DOUBLE Ms,Ps       !  INTEGERS
30    ALLOCATE Q(1:M),R(1:M),S(1:M),A(1:100),G(0:100)
40    M2=M/2.
50    M21=M2-1.
60    U2=U/2.
70    F=1.
80    S=0.
90    FOR Ms=1 TO M
100   Rs=Rs(Ms)
110   T=1./Rs
120   E=Etasq(Ms)
130   Q(Ms)=1.
140   R(Ms)=1.-T
150   S(Ms)=E*T
160   F=F*Rs
170   S=S+E
180   NEXT Ms
190   G(0)=EXP(-.5*S)
200   H=.5*U2^M21*EXP(-U2)/(SQR(F)*FNGamma(M2))
210   Pdf=G(0)*H
220   Q=1.
230   FOR Ps=1 TO 100
240   S=0.
250   FOR Ms=1 TO M
260   R=R(Ms)
270   IF Ps=1 THEN 290
280   Q(Ms)=Q=Q(Ms)*R
290   S=S+Q*(R+Ps*S(Ms))
300   NEXT Ms
310   A(Ps)=.5*S
320   S=0.
330   FOR Ms=1 TO Ps
340   S=S+A(Ms)*G(Ps-Ms)
350   NEXT Ms
360   G(Ps)=G=S/Ps
370   H=H*U2/(M21+Ps)
380   T=G*H
390   Pdf=Pdf+T
400   IF T<Pdf*1.E-15 THEN 440
410   NEXT Ps
420   PRINT "100 TERMS ARE INSUFICIENT"
430   PAUSE
440   RETURN Pdf
450   END
```

MAIN PROGRAM

A program for the evaluation of (D-21), by means of (D-10) and recursions (D-12) and (D-18), is presented below. The desired false alarm and detection probabilities, P_f and P_d respectively, of the processor in figure 1 are specified in lines 10 and 20. Parameter K in line 30 is the number of signal pulses, while M in line 40 is the number of components in fading model (45). The M power ratios $\{r^{(m)}\}$ are inputted in line 50, while the MK deterministic input signal-to-noise ratio measures $\{D_{km}/N_o\}$ are entered in line 60. The K time locations $\{t_k\}$ and frequency locations $\{f_k\}$ of the K transmitted signal pulses are entered in lines 70 and 80. Guesses at the required average random input signal-to-noise ratio measure E_1/N_o are required in lines 90 and 100; the program will search for the required threshold u and required input signal-to-noise ratio E_1/N_o to meet the specifications.

The normalized covariance function of the medium fading is described in function subprogram DEF FNCov(Tau,Nu); currently, it allows for exponential decay in time separation τ and frequency separation ν, but can be easily modified. It must be noted that covariance $R_{mn}(\tau,\nu)$ defined in (47) involves a product of the <u>amplitude</u>-fading quantities $\{g_m(t,f)\}$, not their squares. This distinction, relative to covariance $\tilde{R}_{kj}$ in (55) and normalized covariance ρ_{kj} in (56) between <u>power</u>-fading variates $\{g_m^2(t,f)\}$, must be carefully observed and maintained; furthermore, it is worthwhile to review (66) and (69) at this point, which pertain

only to special cases 5 and 7.

Quantities that need be computed only once and then used repeatedly for different signal-to-noise ratios, such as the eigenvalues and eigenvectors and some auxiliary arrays, are evaluated in the main program and passed in common to function subprogram DEF FNPd for the detection probability. Similarly, for speed and storage purposes, we have identified, for use in this subprogram, the array variables $A(p) = p\ \alpha_p$ for $p \geq 1$ and $G(p) = F\ g_p$ for $p \geq 0$. A separate listing for the false alarm probability P_F in (188) is given in function subprogram DEF FNPf.

The program is listed in BASIC for the Hewlett-Packard 9000 computer. On this particular device, the notation DOUBLE denotes INTEGER variables, not double precision. As a numerical check, the values printed out are threshold U = 38.2583364 and signal-to-noise ratio measure dB = $10 \log_{10}(E_1/N_o)$ = 13.4744256. Since the mean of processor output γ for noise-only is $\overline{\gamma} = 2K = 6$, the normalized threshold is $U/\overline{\gamma} = 6.376$, for this example of P_F = 1E-6.

```
10    Pf=1.E-6                    !  FALSE ALARM PROBABILITY
20    Pd=.9                       !  DETECTION PROBABILITY
30    K=3                         !  NUMBER OF SIGNAL PULSES
40    M=2                         !  NUMBER OF FADING COMPONENTS
50    DATA 1.,1.                  !  POWER RATIOS r(m)   FOR 1 <= m <= M
60    DATA 0,0,0,0,0,0            !  Dkm/No FOR 1 <= k <= K, 1 <= m <= M
70    DATA .1,.3,.5               !  TIMES tk (SEC)      FOR 1 <= k <= K
80    DATA .2,.6,.4               !  FREQUENCIES fk (HZ) FOR 1 <= k <= K
90    E1no0=10.                   !  E1/No STARTING VALUE
100   E1no1=1.                    !  E1/No INCREMENT
110   Ef=1.E-15                   !  TOLERANCE ON Pf
120   Ed=1.E-10                   !  TOLERANCE ON Pd
130   PRINT
140   PRINT "Pf =";Pf;"    K =";K;"    M =";M
150   OPTION BASE 1
160   COM DOUBLE K,M              !  INTEGERS (NOT DOUBLE PRECISION)
170   DOUBLE Ms,Ks,Js             !  INTEGERS
180   DIM Rs(5),Dn(10,5),Ts(10),Fs(10),Psi(5),Cbar(10,10)
190   DIM U(10,10),V(10,10),Eig(10),Sq(10,5)
200   COM U,Prod(5,10),Es(5,10),T(5,10),Q(5,10),S(5,10),B(5,10)
210   COM A(100),G(0:100)
220   REDIM Rs(1:M),Dn(1:K,1:M),Ts(1:K),Fs(1:K),Psi(1:M),Cbar(1:K,1:K)
230   REDIM U(1:K,1:K),V(1:K,1:K),Eig(1:K),Sq(1:K,1:M)
240   REDIM Prod(1:M,1:K),Es(1:M,1:K),T(1:M,1:K)
250   REDIM Q(1:M,1:K),S(1:M,1:K),B(1:M,1:K)
260   READ Rs(*),Dn(*),Ts(*),Fs(*)    !  Dn(*) WAS FILLED IN THE ORDER:
270   S=0.           !  Dn(1,1),Dn(1,2),Dn(1,3),...,Dn(k,m),...,Dn(K,M)
280   FOR Ms=1 TO M
290   S=S+Rs(Ms)
300   NEXT Ms
310   FOR Ms=1 TO M
320   Psi(Ms)=Rs(Ms)/S
330   NEXT Ms
340   FOR Ks=1 TO K
350   FOR Js=1 TO K
360   Cov=FNCov(Ts(Ks)-Ts(Js),Fs(Ks)-Fs(Js))
370   Cbar(Ks,Js)=Cov             !  NORMALIZED COVARIANCE MATRIX
380   NEXT Js
390   NEXT Ks
400   MAT U=Cbar
410   CALL Svd(K,K,U(*),V(*),Eig(*))   !  OUTPUTS: U(*),V(*),Eig(*)
420   PRINT "EIGENVALUES:"
430   PRINT Eig(*)
```

```
440     FOR Ms=1 TO M
450     T=2.*Psi(Ms)
460     FOR Ks=1 TO K
470     Prod(Ms,Ks)=T*Eig(Ks)  !  2 Psi Eig
480     NEXT Ks
490     NEXT Ms
500     FOR Ks=1 TO K
510     FOR Ms=1 TO M
520     Sq(Ks,Ms)=SQR(2.*Dn(Ks,Ms))
530     NEXT Ms
540     NEXT Ks
550     FOR Ks=1 TO K
560     FOR Ms=1 TO M
570     S=0.
580     FOR Js=1 TO K
590     S=S+V(Js,Ks)*Sq(Js,Ms)
600     NEXT Js
610     Es(Ms,Ks)=S*S           !  emk
620     NEXT Ms
630     NEXT Ks
640     U1=1.                   !  THRESHOLD INCREMENT
650     U0=-LOG(Pf)-U1          !  THRESHOLD STARTING VALUE
660     CALL Inversfunction1(-Pf,Ef,U0,U1,U)
670     PRINT "THRESHOLD U =";U
680     CALL Inversfunction2(Pd,Ed,E1no0,E1no1,E1no)
690     Db=10.*LGT(E1no)
700     PRINT "dB =";Db
710     END
720     !
730     DEF FNCov(Tau,Nu)       !  NORMALIZED COVARIANCE
740     A=LOG(2.)*.5            !  R11(Tau,Nu)/R11(0,0)
750     B=1.3
760     Cov=EXP(-A*ABS(Tau)-B*ABS(Nu))
770     RETURN Cov
780     FNEND
790     !
800     DEF FNPf(U)             !  PROBABILITY OF FALSE ALARM
810     COM DOUBLE K            !  INTEGER
820     DOUBLE Ks
830     U2=U*.5
840     Pf=T=EXP(-U2)
850     FOR Ks=1 TO K-1
860     T=T*U2/Ks
870     Pf=Pf+T
880     NEXT Ks
890     RETURN -Pf              !  - TO YIELD INCREASING FUNCTION
900     FNEND
910     !
```

```
920     DEF FNPd(E1no)
930     COM DOUBLE K,M           !  INTEGERS
940     COM U,Prod(*),Es(*),T(*),Q(*),S(*),B(*),A(*),G(*)
950     DOUBLE Ms,Ks,K1,P
960     Tol=1.E-10               !  RELATIVE ERROR OF SUM
970     FOR Ms=1 TO M
980     FOR Ks=1 TO K
990     T(Ms,Ks)=E1no*Prod(Ms,Ks)
1000    NEXT Ks
1010    NEXT Ms
1020    F=1.
1030    S=0.
1040    FOR Ms=1 TO M
1050    FOR Ks=1 TO K
1060    T=T(Ms,Ks)
1070    Q(Ms,Ks)=Q=1.+T
1080    S(Ms,Ks)=T/Q
1090    F=F*Q
1100    S=S+Es(Ms,Ks)/Q
1110    NEXT Ks
1120    NEXT Ms
1130    G(0)=EXP(-.5*S)/SQR(F)
1140    K1=K-1
1150    U2=U*.5
1160    T=EXP(-U2)
1170    H1=1.-T
1180    FOR Ks=1 TO K1
1190    T=T*U2/Ks
1200    H1=H1-T
1210    NEXT Ks
1220    Pd=1.-G(0)*H1
1230    FOR P=1 TO 100
1240    T=T*U2/(K1+P)
1250    H1=H1-T
1260    S=0.
1270    FOR Ms=1 TO M
1280    FOR Ks=1 TO K
1290    Q=Q(Ms,Ks)
1300    IF P>1 THEN 1330
1310    B(Ms,Ks)=B=1./Q
1320    GOTO 1340
1330    B(Ms,Ks)=B=B(Ms,Ks)*S(Ms,Ks)
1340    S=S+B*(T(Ms,Ks)/P+Es(Ms,Ks)/Q)
1350    NEXT Ks
1360    NEXT Ms
1370    A(P)=.5*S*P
1380    S=0.
1390    FOR Ms=1 TO P
1400    S=S+A(Ms)*G(P-Ms)
1410    NEXT Ms
1420    G(P)=G=S/P
1430    Del=G*H1
1440    Pd=Pd-Del
1450    IF Del<Pd*Tol THEN 1490
1460    NEXT P
1470    PRINT "100 TERMS ARE INSUFFICIENT"
1480    PAUSE
1490    PRINT "Pd =";Pd,"    P =";P;"     E1/No =";E1no
1500    RETURN Pd
1510    FNEND
1520    !
```

```
1530    SUB Inversfunction1(Desired,Error,X1,Del,X2)
1540    X2=X1+Del
1550    F1=FNPf(X1)
1560    F2=FNPf(X2)
1570    IF F2>=Desired THEN 1620
1580    X1=X2
1590    X2=X2+Del
1600    F1=F2
1610    GOTO 1560
1620    IF F1<=Desired THEN 1680
1630    X2=X1
1640    X1=X1-Del
1650    F2=F1
1660    F1=FNPf(X1)
1670    GOTO 1620
1680    Xa=X1
1690    Xb=X2
1700    IF F2-Desired<Desired-F1 THEN 1770
1710    T=X1
1720    X1=X2
1730    X2=T
1740    T=F1
1750    F1=F2
1760    F2=T
1770    IF ABS(F2-Desired)<Error THEN 1870
1780    IF F2=F1 THEN 1870
1790    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
1800    T=MAX(T,Xa)
1810    T=MIN(T,Xb)
1820    X1=X2
1830    X2=T
1840    F1=F2
1850    F2=FNPf(X2)
1860    GOTO 1770
1870    SUBEND
1880    !
1890    SUB Inversfunction2(Desired,Error,X1,Del,X2)
1900    X2=X1+Del
1910    F1=FNPd(X1)
1920    F2=FNPd(X2)
1930    IF F2>=Desired THEN 1980
1940    X1=X2
1950    X2=X2+Del
1960    F1=F2
1970    GOTO 1920
1980    IF F1<=Desired THEN 2040
1990    X2=X1
2000    X1=X1-Del
```

```
2010    F2=F1
2020    F1=FNPd(X1)
2030    GOTO 1980
2040    Xa=X1
2050    Xb=X2
2060    IF F2-Desired<Desired-F1 THEN 2130
2070    T=X1
2080    X1=X2
2090    X2=T
2100    T=F1
2110    F1=F2
2120    F2=T
2130    IF ABS(F2-Desired)<Error THEN 2230
2140    IF F2=F1 THEN 2230
2150    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2160    T=MAX(T,Xa)
2170    T=MIN(T,Xb)
2180    X1=X2
2190    X2=T
2200    F1=F2
2210    F2=FNPd(X2)
2220    GOTO 2130
2230    SUBEND
2240    !
2250    SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
2260  ! THIS SUBROUTINE COMPUTES THE SINGULAR VALUE DECOMPOSITION
2270  ! OF AN ARBITRARY REAL MxN MATRIX A:  A = U W Vt, M >= N.
2280  ! U IS MxN, V IS NxN, W IS NxN:  W = DIAG(D(n)).
2290    ALLOCATE Rv1(1:N)              !  NUMERICAL RECIPES, PAGES 60-64
2300    IF M>=N THEN 2330              !  A(*) IS OVER-WRITTEN
2310    PRINT "M<N IS DISALLOWED"
2320    PAUSE
2330    DOUBLE I,J,K,L,Its,Nm,Jj       !  INTEGERS (NOT DOUBLE PRECISION)
2340    G=Scale=Anorm=0.
2350    FOR I=1 TO N
2360    L=I+1
2370    Rv1(I)=Scale*G
2380    G=S=Scale=0.
2390    IF I>M THEN 2670
2400    FOR K=I TO M
2410    Scale=Scale+ABS(A(K,I))
2420    NEXT K
2430    IF Scale=0. THEN 2670
2440    FOR K=I TO M
2450    Aa=A(K,I)=A(K,I)/Scale
2460    S=S+Aa*Aa
2470    NEXT K
2480    F=A(I,I)
2490    G=-SQR(S)
2500    IF F<0. THEN G=-G
```

```
2510    H=F*G-S
2520    A(I,I)=F-G
2530    IF I=N THEN 2640
2540    FOR J=L TO N
2550    S=0.
2560    FOR K=I TO M
2570    S=S+A(K,I)*A(K,J)
2580    NEXT K
2590    F=S/H
2600    FOR K=I TO M
2610    A(K,J)=A(K,J)+F*A(K,I)
2620    NEXT K
2630    NEXT J
2640    FOR K=I TO M
2650    A(K,I)=A(K,I)*Scale
2660    NEXT K
2670    W(I)=Scale*G
2680    G=S=Scale=0.
2690    IF (I>M) OR (I=N) THEN 2990
2700    FOR K=L TO N
2710    Scale=Scale+ABS(A(I,K))
2720    NEXT K
2730    IF Scale=0. THEN 2990
2740    FOR K=L TO N
2750    Aa=A(I,K)=A(I,K)/Scale
2760    S=S+Aa*Aa
2770    NEXT K
2780    F=A(I,L)
2790    G=-SQR(S)
2800    IF F<0. THEN G=-G
2810    H=F*G-S
2820    A(I,L)=F-G
2830    FOR K=L TO N
2840    Rv1(K)=A(I,K)/H
2850    NEXT K
2860    IF I=M THEN 2960
2870    FOR J=L TO M
2880    S=0.
2890    FOR K=L TO N
2900    S=S+A(J,K)*A(I,K)
2910    NEXT K
2920    FOR K=L TO N
2930    A(J,K)=A(J,K)+S*Rv1(K)
2940    NEXT K
2950    NEXT J
2960    FOR K=L TO N
2970    A(I,K)=A(I,K)*Scale
2980    NEXT K
2990    Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
3000    NEXT I
```

```
3010    FOR I=N TO 1 STEP -1
3020    IF I>=N THEN 3190
3030    IF G=0. THEN 3160
3040    FOR J=L TO N
3050    V(J,I)=A(I,J)/A(I,L)/G
3060    NEXT J
3070    FOR J=L TO N
3080    S=0.
3090    FOR K=L TO N
3100    S=S+A(I,K)*V(K,J)
3110    NEXT K
3120    FOR K=L TO N
3130    V(K,J)=V(K,J)+S*V(K,I)
3140    NEXT K
3150    NEXT J
3160    FOR J=L TO N
3170    V(I,J)=V(J,I)=0.
3180    NEXT J
3190    V(I,I)=1.
3200    G=RV1(I)
3210    L=I
3220    NEXT I
3230    FOR I=N TO 1 STEP -1
3240    L=I+1
3250    G=W(I)
3260    IF I>=N THEN 3300
3270    FOR J=L TO N
3280    A(I,J)=0.
3290    NEXT J
3300    IF G=0. THEN 3470
3310    G=1./G
3320    IF I=N THEN 3430
3330    FOR J=L TO N
3340    S=0.
3350    FOR K=L TO M
3360    S=S+A(K,I)*A(K,J)
3370    NEXT K
3380    F=S/A(I,I)*G
3390    FOR K=I TO M
3400    A(K,J)=A(K,J)+F*A(K,I)
3410    NEXT K
3420    NEXT J
3430    FOR J=I TO M
3440    A(J,I)=A(J,I)*G
3450    NEXT J
3460    GOTO 3500
3470    FOR J=I TO M
3480    A(J,I)=0.
3490    NEXT J
3500    A(I,I)=A(I,I)+1.
```

```
3510    NEXT I
3520    FOR K=N TO 1 STEP -1
3530    FOR Its=1 TO 30
3540    FOR L=K TO 1 STEP -1
3550    Nm=L-1
3560    IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3780
3570    IF (ABS(W(Nm))+Anorm)=Anorm THEN 3590
3580    NEXT L
3590    C=0.
3600    S=1.
3610    FOR I=L TO K
3620    F=S*Rv1(I)
3630    Rv1(I)=C*Rv1(I)
3640    IF (ABS(F)+Anorm)=Anorm THEN 3780
3650    G=W(I)
3660    H=SQR(F*F+G*G)
3670    W(I)=H
3680    H=1./H
3690    C=G*H
3700    S=-F*H
3710    FOR J=1 TO M
3720    Y=A(J,Nm)
3730    Z=A(J,I)
3740    A(J,Nm)=Y*C+Z*S
3750    A(J,I)=-Y*S+Z*C
3760    NEXT J
3770    NEXT I
3780    Z=W(K)
3790    IF L<>K THEN 3860
3800    IF Z>=0. THEN 3850
3810    W(K)=-Z
3820    FOR J=1 TO N
3830    V(J,K)=-V(J,K)
3840    NEXT J
3850    GOTO 4390
3860    IF Its<30 THEN 3890
3870    PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3880    PAUSE
3890    X=W(L)
3900    Nm=K-1
3910    Y=W(Nm)
3920    G=Rv1(Nm)
3930    H=Rv1(K)
3940    F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3950    G=SQR(F*F+1.)
3960    Aa=ABS(G)
3970    IF F<0. THEN Aa=-Aa
3980    F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3990    C=S=1.
4000    FOR J=L TO Nm
```

```
4010    I=J+1
4020    G=Rv1(I)
4030    Y=W(I)
4040    H=S*G
4050    G=C*G
4060    Z=SQR(F*F+H*H)
4070    Rv1(J)=Z
4080    C=F/Z
4090    S=H/Z
4100    F=X*C+G*S
4110    G=-X*S+G*C
4120    H=Y*S
4130    Y=Y*C
4140    FOR Jj=1 TO N
4150    X=V(Jj,J)
4160    Z=V(Jj,I)
4170    V(Jj,J)=X*C+Z*S
4180    V(Jj,I)=-X*S+Z*C
4190    NEXT Jj
4200    Z=SQR(F*F+H*H)
4210    W(J)=Z
4220    IF Z=0. THEN 4260
4230    Z=1./Z
4240    C=F*Z
4250    S=H*Z
4260    F=C*G+S*Y
4270    X=-S*G+C*Y
4280    FOR Jj=1 TO M
4290    Y=A(Jj,J)
4300    Z=A(Jj,I)
4310    A(Jj,J)=Y*C+Z*S
4320    A(Jj,I)=-Y*S+Z*C
4330    NEXT Jj
4340    NEXT J
4350    Rv1(L)=0.
4360    Rv1(K)=F
4370    W(K)=X
4380    NEXT Its
4390    NEXT K
4400    SUBEND
```

APPENDIX E. SOLUTION OF GENERALIZED EIGENVALUE PROBLEM

C and B are N×N real symmetric matrices, and B is positive definite. The important alternative case where B is not positive definite will be undertaken in (E-14) and sequel. We want the normalized modal matrix Q and the eigenvalue matrix Λ of the generalized characteristic-value problem [18; page 74] encountered in (B-10), namely

$$C\ Q = B^{-1}\ Q\ \Lambda\ . \tag{E-1}$$

Then, it is known that [18; pages 74 - 77]

$$Q^T\ B^{-1}\ Q = I \quad \text{and} \quad Q^T\ C\ Q = \Lambda\ , \tag{E-2}$$

which is a simultaneous reduction of two matrices to diagonal form. Alternatively, when both equations in (E-2) are satisfied, then (E-1) follows.

The first relation in (E-2) sets the scalings on the eigenvectors $\{V_n\}$ of Q; in fact, it reads $V_n^T\ B^{-1}\ V_n = 1$ for $1 \le n \le N$. An alternative scaling choice would be to make the eigenvectors of unit length, that is, $V_n^T\ V_n = 1$. However, this would result in $Q^T\ B^{-1}\ Q$ becoming a diagonal matrix with non-unity elements; this latter alternative is not adopted here. Therefore, the eigenvectors $\{V_n\}$ in (E-2) do not have unit length, that is, $V_n^T\ V_n \ne 1$ generally. The eigenvectors are unique within their polarities.

A procedure for determining Q and Λ is now presented. Solve for the square root matrix S in

$$B = S\ S^T\ , \tag{E-3}$$

where S need not be symmetric. For example, the lower (or upper) triangular square root matrix will suffice. Then there follows

$$B^{-1} = S^{-T} S^{-1} \quad \text{and} \quad S^{T} B^{-1} S = I \ . \tag{E-4}$$

Now compute N×N matrix

$$A = S^{T} C S \ , \tag{E-5}$$

which is symmetric. Solve the <u>standard</u> eigenvalue problem

$$A U = U \gamma \tag{E-6}$$

for normalized modal matrix U and diagonal eigenvalue matrix γ. Then it is known that these solutions satisfy

$$U^{T} U = I \quad \text{and} \quad U^{T} A U = \gamma \ . \tag{E-7}$$

It will now be shown that

$$Q = S U \quad \text{and} \quad \Lambda = \gamma \tag{E-8}$$

are the desired matrix solutions to (E-1). Substitution of (E-8) in the first relation in (E-2) yields

$$Q^{T} B^{-1} Q = U^{T} S^{T} B^{-1} S U = U^{T} I U = I \ , \tag{E-9}$$

where we used (E-4) and (E-7). Also, substitution of (E-8) in the second relation in (E-2) yields

$$Q^{T} C Q - \Lambda = U^{T} S^{T} C S U - \gamma = U^{T} A U - \gamma = 0 \ , \tag{E-10}$$

where we used (E-5) and (E-7). Thus, both relations in (E-2) are satisfied by the matrix assignments in (E-8).

In summary, in order to solve (E-1): compute square root matrix S in (E-3); evaluate matrix A via (E-5); solve (E-6) for U and γ; determine Q and Λ by means of (E-8). Notice there is never any need to compute inverse matrix B^{-1}; however, matrix B must be positive definite for the operation in (E-3) to succeed.

We now present an example which illustrates many of the quantities considered above; let matrices

$$B = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}. \tag{E-11}$$

Then the solutions of (E-1) are

$$\Lambda = \begin{bmatrix} 20 & 0 \\ 0 & 12 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}. \tag{E-12}$$

The column vectors of Q are not orthogonal; in fact,

$$Q^T Q = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}. \tag{E-13}$$

Rather, the orthogonality relations of (E-2) may be readily verified to be satisfied in this example, namely $Q^T B^{-1} Q = I$ and $Q^T C Q = \Lambda$.

GENERAL SYMMETRIC MATRIX B

In attempting to solve (E-1) for Q and Λ, the square root of matrix B was required in (E-3). If B is not positive definite, complex solutions would be required there. Here, we will solve the alternative generalized characteristic-value matrix equation encountered in (B-44), namely

$$B\,Q = C^{-1}\,Q\,\Lambda\,, \tag{E-14}$$

where covariance matrix C is real symmetric and positive definite, while matrix B is real symmetric but can be indefinite. The solutions Q and Λ will then satisfy the relations

$$Q^T\,C^{-1}\,Q = I \quad \text{and} \quad Q^T\,B\,Q = \Lambda\,. \tag{E-15}$$

To this aim, first solve for the square root matrix S in

$$C = S\,S^T\,. \tag{E-16}$$

Then

$$C^{-1} = S^{-T}\,S^{-1} \quad \text{and} \quad S^T\,C^{-1}\,S = I\,. \tag{E-17}$$

Next, compute the N×N matrix

$$A = S^T\,B\,S\,, \tag{E-18}$$

which is symmetric. Solve the standard characteristic-value equation

$$A\,U = U\,\gamma \tag{E-19}$$

for U and γ, for which we have orthogonality properties

$$U^T U = I \quad \text{and} \quad U^T A U = \gamma . \tag{E-20}$$

It will now be shown that the solutions to (E-14) are

$$Q = S U \quad \text{and} \quad \Lambda = \gamma . \tag{E-21}$$

Substitution of (E-21) in the first relation in (E-15) yields

$$Q^T C^{-1} Q = U^T S^T C^{-1} S U = U^T I U = U^T U = I , \tag{E-22}$$

where we used (E-17) and (E-20). Similarly, substitution of (E-21) in the second relation in (E-15) yields

$$Q^T B Q - \Lambda = U^T S^T B S U - \gamma = U^T A U - \gamma = 0 , \tag{E-23}$$

where we used (E-18) and (E-20). Thus, both relations in (E-15) are verified by solutions (E-21).

In summary, in order to solve (E-14): compute square root matrix S in (E-16); evaluate matrix **A** according to (E-18); solve (E-19) for **U** and γ; determine Q and Λ by means of (E-21). This procedure requires covariance matrix C to be positive definite. Inverses of matrices C, B, or S are not required.

(If square root matrix S in (E-16) is taken lower triangular, then S^{-1} is also lower triangular and can be very easily computed directly from S. This enables calculation of inverse matrix C^{-1} by means of (E-17), if desired.)

INTERRELATIONSHIPS OF SOLUTIONS

Here we shall relate the solutions Q and Λ of (E-1) to the solutions Q and Λ of (E-14). In particular, we maintain that the specific relations are

$$Q = Q^{-T}\,\Lambda^{\frac{1}{2}} \quad \text{and} \quad \Lambda = \Lambda\ . \tag{E-24}$$

To verify this claim, substitute (E-24) into the first relation in (E-2) and obtain

$$Q^T\,B^{-1}\,Q - I = \Lambda^{\frac{1}{2}}\,Q^{-1}\,B^{-1}\,Q^{-T}\,\Lambda^{\frac{1}{2}} - I = \Lambda^{\frac{1}{2}}\,\Lambda^{-1}\,\Lambda^{\frac{1}{2}} - I = 0\ , \tag{E-25}$$

where we used (E-15). Similarly, substitution in the second relation in (E-2) yields

$$Q^T\,C\,Q - \Lambda = \Lambda^{\frac{1}{2}}\,Q^{-1}\,C\,Q^{-T}\,\Lambda^{\frac{1}{2}} - \Lambda = \Lambda^{\frac{1}{2}}\,I\,\Lambda^{\frac{1}{2}} - \Lambda = 0\ , \tag{E-26}$$

where we also used (E-15). Thus, both relations in (E-2) are satisfied by interrelationships (E-24).

The connection between eigenvectors is obtained by utilizing (E-15) to give

$$Q^{-T} = B\,Q\,\Lambda^{-1}\ . \tag{E-27}$$

Substitution in (E-24) then yields

$$Q = B\,Q\,\Lambda^{-\frac{1}{2}}\ ; \quad \text{that is,} \quad V_n = B\,V_n\,\lambda_n^{-\frac{1}{2}}\ . \tag{E-28}$$

The eigenvalue relation is, directly from (E-24),

$$\lambda_n = \lambda_n\ . \tag{E-29}$$

Finally, the connection between vectors E and A in (B-18) and the corresponding vectors E and A in (B-51) and (B-52) is as follows: the use of (E-28) in the first relation in (B-18) yields

$$\varepsilon_n = V_n^T E = \lambda_n^{-1/2} V_n^T B E = \lambda_n^{1/2} \varepsilon_n , \qquad \text{(E-30)}$$

the last relation based on (B-59). And substitution of (E-28) in the second relation in (B-18) yields

$$\alpha_n = V_n^T B^{-1} A = \lambda_n^{-1/2} V_n^T B B^{-1} A = \lambda_n^{-1/2} \alpha_n , \qquad \text{(E-31)}$$

using (B-60).

It can now be seen that the use of (E-29) - (E-31) in characteristic function (B-20) immediately converts it to (B-54). This is a direct confirmation of the fact that the characteristic function of q is unique, regardless of how derived. However, the recommended procedure is that given in (B-43) and sequel, because it is more general, allowing symmetric matrix B to be indefinite. In fact, relation (E-28) is valid only when all eigenvalues λ_n are positive. If this is not the case, although Q cannot be found using real arithmetic, the technique summarized under (E-23), for determining solutions Q and Λ of (E-14), is still valid and can always be used with real arithmetic. Eigenvalues $\{\lambda_n\}$ can take on positive and negative values. The only restriction is that symmetric matrix C be positive definite.

APPENDIX F. STATISTICS OF REAL BILINEAR FORM

Suppose we encounter real bilinear form [22; (13.5-1)]

$$b = X^T A Y , \tag{F-1}$$

where random vector X is M×1, random vector Y is N×1, and arbitrary matrix A is M×N, all quantities being real. We allow random vectors X and Y to be arbitrarily correlated with each other, but to have zero means here; generalizations to nonzero means can be made along the lines of appendix B.

Let L = M+N and define L×1 real vector

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix} . \tag{F-2}$$

Let L×L matrix

$$B = \frac{1}{2} \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} = B^T . \tag{F-3}$$

Then (F-1) becomes

$$b = Z^T B Z , \tag{F-4}$$

which is a quadratic form in L variables; L = M+N. Matrix B is symmetric but not positive definite. Letting r = min(M,N), numerical investigation has revealed that B has r positive eigenvalues $\mu_1,\ldots,\mu_r$; r eigenvalues which are their negatives $-\mu_1,\ldots,-\mu_r$; and the rest zero.

The mean of real random vector Z is zero, and its (arbitrary) L×L covariance matrix is

$$C = \overline{Z\ Z^T} = \overline{\begin{bmatrix} X \\ Y \end{bmatrix} [X^T \quad Y^T]} = \begin{bmatrix} \overline{X\ X^T} & \overline{X\ Y^T} \\ \overline{Y\ X^T} & \overline{Y\ Y^T} \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy}^T & C_{yy} \end{bmatrix} , \quad \text{(F-5)}$$

which is symmetric. C is also nonnegative definite, because

$$V^T\ C\ V = V^T\ \overline{Z\ Z^T}\ V = \overline{\left(V^T\ Z\right)^2} \geq 0 \quad \text{(F-6)}$$

for any L×1 vector V. We presume that C is positive definite.

In order to convert quadratic form (F-4) to diagonal form with uncorrelated variates, we first solve the L×L generalized characteristic-value equation (see (E-14) and sequel)

$$B\ Q = C^{-1}\ Q\ \Lambda \quad \text{(F-7)}$$

for L×L matrices Q and Λ. Then, we have properties [18; pages 74 - 77]

$$Q^T\ C^{-1}\ Q = I \quad \text{and} \quad Q^T\ B\ Q = \Lambda = \mathrm{diag}[\lambda_1\ .\ .\ .\ \lambda_L]\ . \quad \text{(F-8)}$$

Now let linearly transformed random vector

$$W = Q^{-1}\ Z = [w_1\ .\ .\ .\ w_L] \quad ; \quad Z = Q\ W\ . \quad \text{(F-9)}$$

Vector W also has zero mean. Its covariance matrix is

$$\overline{W\ W^T} = Q^{-1}\ \overline{Z\ Z^T}\ Q^{-T} = Q^{-1}\ C\ Q^{-T} = I\ , \quad \text{(F-10)}$$

upon use of (F-5) and (F-8). Also, bilinear form (F-4) becomes

$$b = Z^T\ B\ Z = W^T\ Q^T\ B\ Q\ W = W^T\ \Lambda\ W = \sum_{k=1}^{L} \lambda_k w_k^2\ , \quad \text{(F-11)}$$

where we used (F-9) and (F-8). Thus, b is a sum of L weighted squares of uncorrelated zero-mean unit-variance random variables. Notice that not all eigenvalues $\{\lambda_k\}$ need be positive; some can be zero and some can be negative.

If random vectors X and Y are joint-Gaussian, then vector W in (F-9) is also Gaussian, with probability density function

$$p(W) = \prod_{k=1}^{L} \left((2\pi)^{-\frac{1}{2}} \exp(-w_k^2/2) \right) , \tag{F-12}$$

based on (F-10). The characteristic function of bilinear form b in (F-11) is then

$$f_b(\xi) = \overline{\exp(i\xi b)} = \prod_{k=1}^{L} \left(\int dw_k \; (2\pi)^{-\frac{1}{2}} \exp\left[- w_k^2/2 + i\xi\lambda_k w_k^2 \right] \right) =$$

$$= \left[\prod_{k=1}^{L} \left(1 - i2\xi\lambda_k \right) \right]^{-\frac{1}{2}} . \tag{F-13}$$

Since only the eigenvalues $\{\lambda_k\}$ of Λ in (F-8) are required to evaluate characteristic function (F-13), it is not actually necessary to determine the eigenvectors Q in matrix equation (F-7). In fact, by a procedure identical to that given earlier in (B-55) - (B-58), it may be shown that $\{\lambda_k\}$ are the eigenvalues of nonsymmetric L×L matrix B C. Reference to (F-3) and (F-5) reveals that this latter matrix is given by

$$B\,C = \frac{1}{2} \begin{bmatrix} A\,C_{xy}^T & A\,C_{yy} \\ A^T\,C_{xx} & A^T\,C_{xy} \end{bmatrix} . \tag{F-14}$$

The rank of matrix B C is generally $R = 2\min(M,N)$; that is, the product for characteristic function $f_b(\xi)$ in (F-13) will contain only R terms, which is less than L except when $M = N$.

The claim for the rank of B C is based upon the following observation. Suppose that $M \le N$; then (F-1) can be written as

$$b = \sum_{m=1}^{M} \sum_{n=1}^{N} x_m \, a_{mn} \, y_n = \sum_{m=1}^{M} x_m \, v_m \, , \tag{F-15}$$

where (linearly transformed) Gaussian random variables

$$v_m = \sum_{n=1}^{N} a_{mn} \, y_n \quad \text{for } 1 \le m \le M \, . \tag{F-16}$$

Thus, b is a sum of products of just 2M correlated random variables. Therefore, the characteristic function in (F-13) can have no more than 2M terms. Direct numerical evaluation of the eigenvalues $\{\lambda_k\}$ of nonsymmetric L×L matrix B C in (F-14) has verified that its rank is generally $R = 2\min(M,N)$, and that the R nonzero eigenvalues can be positive and negative with no obvious interrelations. (If $N < M$, the summation order in (F-15) can be reversed, thereby giving b as a sum of products of 2N correlated random variables, and hence, rank 2N for matrix B C.)

The generalization of (F-1) to the form

$$b = X^T A_{11} X + X^T A_{12} Y + Y^T A_{21} X + Y^T A_{22} Y \tag{F-17}$$

can be directly fit into the framework of (F-2) and (F-4) if we generalize definition (F-3) to L×L matrix

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} . \qquad \text{(F-18)}$$

We can then replace this B matrix by its symmetric part, since only the symmetric part of B is active in quadratic form (F-4).

APPENDIX G. CHARACTERISTIC FUNCTION OF MOST GENERAL COMPLEX FORM WITH FIRST-ORDER AND SECOND-ORDER TERMS

Let Z be a complex N×1 random vector with real and imaginary parts X and Y; that is, $Z = X + i\,Y$. Let D_1 and D_2 be complex N×1 constant vectors. Let C_1, C_2, C_3, C_4 be complex N×N constant matrices, which need not be Hermitian or symmetric.

The most general first-order complex form is

$$f_1 = D_1^T\, Z + D_2^T\, Z^* = (D_1 + D_2)^T\, X + i(D_1 - D_2)^T\, Y = H^T\, W\ , \quad \text{(G-1)}$$

where

$$H = \begin{bmatrix} D_1 + D_2 \\ i(D_1 - D_2) \end{bmatrix}, \qquad W = \begin{bmatrix} X \\ Y \end{bmatrix}. \quad \text{(G-2)}$$

H is a complex 2N×1 constant vector and is completely arbitrary; that is, every complex element of H can be independently specified. W is a real 2N×1 random vector.

The most general second-order complex form is

$$f_2 = Z^H\, C_1\, Z + Z^H\, C_2\, Z^* + Z^T\, C_3\, Z + Z^T\, C_4\, Z^* = W^T\, M\, W\ , \quad \text{(G-3)}$$

where

$$M = \begin{bmatrix} C_1 + C_2 + C_3 + C_4 & i(C_1 - C_2 + C_3 - C_4) \\ -i(C_1 + C_2 - C_3 - C_4) & C_1 - C_2 - C_3 + C_4 \end{bmatrix}. \quad \text{(G-4)}$$

M is a complex 2N×2N constant matrix, which need not be Hermitian. M is completely arbitrary; that is, every complex element of M can be independently specified.

The pertinent statistics of real vector W are

$$\overline{W} = \begin{bmatrix} \overline{X} \\ \overline{Y} \end{bmatrix} , \quad K = \mathrm{Cov}(W) = \overline{(W - \overline{W})\ (W - \overline{W})^T} =$$

$$= \overline{\begin{bmatrix} X - \overline{X} \\ Y - \overline{Y} \end{bmatrix} \begin{bmatrix} X^T - \overline{X}^T & Y^T - \overline{Y}^T \end{bmatrix}} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} . \qquad \text{(G-5)}$$

Here, $\overline{W}$ is a real 2N×1 constant vector, while K is a real 2N×2N symmetric constant matrix.

The general complex form of interest is

$$c = f_2 + f_1 = W^T\, M\, W + H^T\, W . \qquad \text{(G-6)}$$

For X and Y joint-Gaussian, W is Gaussian with probability density function

$$p(W) = (2\pi)^{-N}\ (\det\ K)^{-\frac{1}{2}}\ \exp\left(-\ \frac{1}{2}(W - \overline{W})^T\ K^{-1}\ (W - \overline{W})\right) . \qquad \text{(G-7)}$$

The statistical quantity of interest is, for complex scalar α, the characteristic function of complex random variable c in (G-6); in particular, the characteristic function is defined here as the average of the exponential

$$f_c(\alpha) = \overline{\exp(\alpha c)} = \overline{\exp\left(\alpha\ W^T\ M\ W + \alpha\ H^T\ W\right)} = (2\pi)^{-N}\ (\det\ K)^{-\frac{1}{2}}\ \times$$

$$\times \int dW\ \exp\left(-\ \frac{1}{2}(W - \overline{W})^T\ K^{-1}\ (W - \overline{W}) + \alpha\ W^T\ M\ W + \alpha\ H^T\ W\right) =$$

$$= d^{-\frac{1}{2}}\ \exp(t) , \qquad \text{(G-8)}$$

where

$$d = \det(I - 2\,\alpha\, M\, K) = \det(I - 2\,\alpha\, K\, M)\ ,$$

$$t = -\frac{1}{2}\,\overline{W}^T\, K^{-1}\,\overline{W} + \frac{1}{2}\, V^T\,(I - 2\,\alpha\, K\, M)^{-1}\, K\, V\ ,$$

$$V = K^{-1}\,\overline{W} + \alpha\, H\ . \tag{G-9}$$

K is a real 2N×2N matrix, M is a complex 2N×2N matrix which need not be Hermitian, $\overline{W}$ is a real 2N×1 vector, H and V are complex 2N×1 vectors, and α is a complex scalar.

In general, we must invert 2N×2N real symmetric matrix K. Also, we must invert 2N×2N complex matrix $I - 2\,\alpha\, K\, M$, which need not be Hermitian, and which depends on argument α. The average of the exponential in (G-8) is the type of operation encountered in appendix C.

If $\overline{X} = 0$, $\overline{Y} = 0$, $D_1 = 0$, $D_2 = 0$, then $\overline{W} = 0$, $H = 0$, $V = 0$, $t = 0$, and we need only evaluate complex determinant $d = \det(I - 2\,\alpha\, K\, M)$, which depends on α.

SPECIAL CASE: $Y = 0$

Then complex forms

$$f_1 = (D_1 + D_2)^T X \equiv D^T X ,$$

$$f_2 = X^T (C_1 + C_2 + C_3 + C_4) X \equiv X^T C X ,$$

$$c = X^T C X + D^T X . \tag{G-10}$$

Matrices C and D are complex and completely arbitrary.

Identify in the subsection above,

$$2N \rightarrow N , \quad W \rightarrow X , \quad M \rightarrow C , \quad H \rightarrow D , \quad K \rightarrow K_{xx} , \tag{G-11}$$

thereby getting characteristic function

$$f_c(\alpha) = \overline{\exp(\alpha c)} = d^{-\frac{1}{2}} \exp(t) , \tag{G-12}$$

where

$$d = \det(I - 2 \alpha K_{xx} C) ,$$

$$t = -\frac{1}{2} \overline{X}^T K_{xx}^{-1} \overline{X} + \frac{1}{2} V^T (I - 2 \alpha K_{xx} C)^{-1} K_{xx} V ,$$

$$V = K_{xx}^{-1} \overline{X} + \alpha D . \tag{G-13}$$

K_{xx} is a real N×N matrix, C is a complex N×N matrix which need not be Hermitian, $\overline{X}$ is a real N×1 vector, D and V are complex N×1 vectors, and α is a complex scalar.

JOINT CHARACTERISTIC FUNCTION OF REAL AND IMAGINARY PARTS OF c

For the general complex form c in (G-6), the joint characteristic function of c_r and c_i is, for real ξ and ζ,

$$f(\xi,\zeta) \equiv \overline{\exp(i\xi\, c_r + i\zeta\, c_i)}\ . \tag{G-14}$$

But

$$\xi\, c_r + \zeta\, c_i = \xi\left(W^T M_r W + H_r^T W\right) + \zeta\left(W^T M_i W + H_i^T W\right) =$$
$$= W^T \tilde{M} W + \tilde{H}^T W\ , \tag{G-15}$$

where real matrices

$$\tilde{M} \equiv \xi\, M_r + \zeta\, M_i\ , \quad \tilde{H} \equiv \xi\, H_r + \zeta\, H_i\ . \tag{G-16}$$

Therefore, identifying $\alpha \to i$, $M \to \tilde{M}$, $H \to \tilde{H}$, in (G-8), there follows the joint characteristic function of c_r and c_i as

$$f(\xi,\zeta) = \overline{\exp\left[i\, W^T \tilde{M} W + i\, \tilde{H}^T W\right]} = d^{-\frac{1}{2}} \exp(t)\ , \tag{G-17}$$

where

$$d = \det(I - i\, 2\, \tilde{M} K) = \det(I - i\, 2\, K \tilde{M})\ ,$$

$$t = -\frac{1}{2} \overline{W}^T K^{-1} \overline{W} + \frac{1}{2} V^T (I - i\, 2\, K \tilde{M})^{-1} K V\ ,$$

$$V = K^{-1} \overline{W} + i\, \tilde{H}\ . \tag{G-18}$$

REFERENCES

[1] J. I. Marcum, A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix, **IRE Transactions on Information Theory**, volume IT-6, number 2, pages 145 - 267, April 1960. Originally presented in Research Memorandum RM-753, RAND Corporation, Santa Monica, CA, 1 July 1948.

[2] P. Swerling, "Probability of Detection of Fluctuating Targets", **IRE Transactions on Information Theory**, volume IT-6, number 2, pages 269 - 308, April 1960. Originally presented in Research Memorandum RM-1217, RAND Corporation, Santa Monica, CA, 17 March 1954.

[3] M. Schwartz, "Effects of Signal Fluctuations in the Detection of Pulsed Signals in Noise", **IRE Transactions on Information Theory**, volume IT-2, number 2, pages 66 - 71, June 1956.

[4] P. Swerling, "Detection of Fluctuating Pulsed Signals in the Presence of Noise", **IRE Transactions on Information Theory**, volume IT-3, number 3, pages 175 - 178, September 1957.

[5] M. Nakagami, The m-Distribution - A General Formula of Intensity Distribution of Rapid Fading, **Statistical Methods in Radio Wave Propagation Proceedings**, UCLA, W. C. Hoffman, editor, Pergamon Press, 1958.

[6] P. Swerling, "Detection of Radar Echoes in Noise Revisited", **IEEE Transactions on Information Theory**, volume IT-12, number 3, pages 348 - 361, July 1966.

[7] M. Schwartz, W. R. Bennett, and S. Stein, **Communication Systems and Techniques**, McGraw-Hill Book Company, Inc., New York, NY, 1966.

[8] G. R. Heidbreder and R. L. Mitchell, "Detection Probabilities for Log-Normally Distributed Signals", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-3, number 1, pages 5 - 13, January 1967.

[9] R. A. Scholtz, J. J. Kappl, and N. E. Nahi, "The Detection of Moderately-Fluctuating Rayleigh Targets", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-12, number 2, pages 117 - 126, March 1976.

[10] A. H. Nuttall, **Operating Characteristics for Detection of a Fading Signal with K Dependent Fades and D-fold Diversity in M Alternative Locations**, NUSC Technical Report 5739, Naval Underwater Systems Center, New London, CT, 25 October 1977.

[11] I. Kanter, "Exact Detection Probability for Partially Correlated Rayleigh Targets", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-22, number 2, pages 184 - 195, March 1986.

[12] A. H. Nuttall and E. S. Eby, **Signal-to-Noise Ratio Requirements for Detection of Multiple Pulses Subject to Partially Correlated Fading With Chi-Squared Statistics of Various Degrees of Freedom**, NUSC Technical Report 7707, Naval Underwater Systems Center, New London, CT, 2 June 1986.

[13] S. Watts, "Radar Detection Prediction in K-Distributed Sea Clutter and Thermal Noise", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-23, number 1, pages 40 - 45, January 1987.

[14] A. H. Nuttall, **Detection Performance of Normalizer for Multiple Signals Subject to Partially Correlated Fading With Chi-Squared Statistics**, NUSC Technical Report 8133, Naval Underwater Systems Center, New London, CT, 1 October 1987.

[15] X. Y. Hou and N. Morinaga, "Detection Performance in K-Distributed and Correlated Rayleigh Clutter", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-25, number 5, pages 634 - 642, September 1989.

[16] I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, Inc., New York, NY, 1980.

[17] A. H. Nuttall, **Evaluation of Densities and Distributions via Hermite and Generalized Laguerre Series Employing High-Order Expansion Coefficients Determined Recursively via Moments or Cumulants**, NUSC Technical Report 7377, Naval Underwater Systems Center, New London, CT, 28 February 1985.

[18] F. B. Hildebrand, **Methods of Applied Mathematics**, Prentice-Hall, Inc., New York, NY, 1954.

[19] C. W. Helstrom, **Statistical Theory of Signal Detection**, second edition, Pergamon Press, Inc., New York, NY, 1968.

[20] W. B. Davenport and W. L. Root, **Random Signals and Noise**, McGraw-Hill Book Company, Inc., New York, NY, 1958.

[21] A. H. Nuttall, **Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions**, NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

[22] G. A. Korn and T. M. Korn, **Mathematical Handbook for Scientists and Engineers**, McGraw-Hill Book Co., Inc., New York, NY, 1961.

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Statistics of a Whiteness Measure

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ABSTRACT

A whiteness measure of a random number generator is defined as the sum of squares of all the off-zero elements of the sample covariance function of a finite segment of data of length K. The mean and variance of this whiteness measure are evaluated exactly, while its cumulative and exceedance distribution functions are determined by simulations. It is found that the variance of the whiteness measure must be broken into the two separate cases where K is even versus K is odd. This necessitates the analytic evaluation of some high-order moments in order to determine the variance exactly.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

K	number of data points, (1)
x_k	k-th data value, (1)
IID	independent identically distributed
E( )	expectation of random variable, (1)
F	fourth moment of random variable x_k, (1)
n	n-th delay, (2)
R_n	sample covariance at delay n, (2)
W_K	whiteness measure, (3)
Var	variance, (10)
V_K	variance of K-th whiteness measure W_K, (10)
A,B,C,D	unknown constants in (10) and (35)
ϕ_n	sum of delayed products of data, (11)
$A,\tilde{B},\tilde{C}$	constants for K odd, (45)
A,B,C	constants for K even, (48)
p(x)	probability density function of random variable x, (51)
D	constant for K even or odd, (55)
M	size of fast Fourier transform, (56)
$\{X_m\}$	fast Fourier transform of data $\{x_k\}$, (56)
u	threshold value, (56)
CDF(u)	cumulative distribution function, (57)
EDF(u)	exceedance distribution function, (57)
V_n	interval probabilities, (B-1)
C	cumulative distribution function, (B-3)
E	exceedance distribution function, (B-3)
μ_n	n-th moment, (B-8)

STATISTICS OF A WHITENESS MEASURE

INTRODUCTION

When a random number generator is designed to yield zero-mean independent random variables, one useful test of its validity is afforded by its sample covariance function. This quantity would ideally be zero for all delays except the origin value. However, in practice, due to the finite length of data generated and used to test the generator, the sample covariance function is not identically zero but fluctuates about zero. A measure of the whiteness of the generator is afforded by the sum of squares of all the off-zero elements of the sample covariance function, relative to the square of its origin value. This measure was suggested in [1; appendix C].

In this report, we investigate the statistics of this whiteness measure, including its cumulative and exceedance distribution functions and its mean and variance. Since a sample covariance involves products of data values, the squared covariance depends on fourth-order products of the data, and the variance of this sample quantity involves eighth-order products of the data under various delays. It is this latter high-order product which greatly complicates the statistical analysis and which necessitates a roundabout procedure for exact evaluation of the variance of the whiteness measure. The probability distributions of this measure are determined by simulation for two types of random variables, uniform and Gaussian.

MEAN AND VARIANCE OF WHITENESS MEASURE

Consider real data sequence x_0, x_1, ... ,x_{K-1} of K data points which are independent and identically distributed (IID) with a symmetric probability density function about zero. This zero-mean sequence will have all odd-order moments equal to zero. Also, assume that the data are scaled to have unit variance and a fourth moment of value F; that is

$$E\left(x_k^2\right) = 1 \ , \quad E\left(x_k^4\right) = F \ , \quad \text{for } 0 \le k \le K-1 \ , \tag{1}$$

where E denotes the expectation. This situation includes the uniform random number generator and the Gaussian random number generator, for example. For the usual uniform random variable distributed over $(-\frac{1}{2},\frac{1}{2})$, we have scaled its output by $\sqrt{12}$ for present purposes in order to realize variance 1. Thus, F = 1.8 for the uniform case, while F = 3 for Gaussian numbers.

The sample covariance of the available data is defined as

$$R_n = \frac{1}{K} \sum_k x_k \, x_{k-n} \quad \text{for all } n \ . \tag{2}$$

Ideally, we might like to have sequence $\{R_n\}$ equal to zero for $n \neq 0$. However, this is never the case, although the $\{R_n\}$ for $n \neq 0$ are much smaller than R_0 when K is large. The mean value of R_0 is easily seen to be 1, by reference to (1). A measure of the whiteness of data sequence $\{x_k\}$ is afforded by the sum of squares of all the off-zero elements of sequence $\{R_n\}$:

$$W_K \equiv \sum_{n \neq 0} R_n^2 = 2 \sum_{n=1}^{K-1} R_n^2 \quad \text{for } K \ge 2 \ . \tag{3}$$

MEAN OF WHITENESS MEASURE W_K

The mean value of random variable R_n^2 follows from (2) as

$$E\left(R_n^2\right) = E\left(\frac{1}{K^2} \sum_k \sum_j x_k\, x_{k-n}\, x_j\, x_{j-n}\right) =$$

$$= \frac{1}{K^2} \sum_k \sum_j E\left(x_k\, x_{k-n}\, x_j\, x_{j-n}\right) . \tag{4}$$

Since we are only interested in values of $n > 0$ according to (3), the expectation in (4) is nonzero only when $k = j$; here, we are utilizing both the IID and the zero-mean properties of $\{x_k\}$. Then, (4) becomes, upon use of (1),

$$E\left(R_n^2\right) = \frac{1}{K^2} \sum_{k=n}^{K-1} 1 = \frac{K - n}{K^2} \quad \text{for } 1 \le n \le K-1 . \tag{5}$$

(For completeness, $E\left(R_0^2\right) = (F + K - 1)/K$; Variance$(R_0) = (F-1)/K$. Thus, R_0 clusters around 1 as $K \to \infty$, while $R_n \to 0$ as $K \to \infty$ for fixed $n \neq 0$.) Use of result (5) in (3) yields the desired mean value of whiteness measure W_K as

$$E(W_K) = \frac{2}{K^2} \sum_{n=1}^{K-1} (K - n) = \frac{K - 1}{K} . \tag{6}$$

Notice that this mean value is independent of fourth-moment F and that it approaches 1 as $K \to \infty$. Recall that $E(R_0) = 1$ for comparison.

VARIANCE OF WHITENESS MEASURE W_K

The direct evaluation of the variance of random variable W_K in (3) would require a very tedious procedure. Whereas the mean evaluation in (4) only encountered fourth-order products of delayed versions of $\{x_k\}$, we would now encounter eighth-order products, requiring a complicated counting procedure to account for all the various types of terms. Specifically, from (2) and (3), we have whiteness measure

$$W_K = \frac{2}{K^2} \sum_{n=1}^{K-1} \sum_{k=n}^{K-1} \sum_{j=n}^{K-1} x_k \, x_{k-n} \, x_j \, x_{j-n} \; , \tag{7}$$

leading to mean square value

$$E\left(W_K^2\right) = \frac{4}{K^4} \sum_{n=1}^{K-1} \sum_{m=1}^{K-1} \sum_{k=n}^{K-1} \sum_{j=n}^{K-1} \sum_{q=m}^{K-1} \sum_{p=m}^{K-1}$$

$$E\left(x_k \, x_{k-n} \, x_j \, x_{j-n} \, x_q \, x_{q-m} \, x_p \, x_{p-m}\right) \; . \tag{8}$$

Not only would this eighth-order average have to be evaluated for all possible values of n,m,k,j,q,p, but the sixth-order summation would then have to be conducted. The only reasonable case that can be evaluated from (8) is that for the term proportional to F^2. It is obtained only for the special choices $n = m$ and $k = j = q = p$; then the right-hand side of (8) reduces to

$$\frac{4}{K^4} \sum_{n=1}^{K-1} \sum_{k=n}^{K-1} F^2 = \frac{4}{K^4} \sum_{n=1}^{K-1} (K - n) \, F^2 = \frac{2(K-1)}{K^3} \, F^2 \; . \tag{9}$$

Notice that moments of $\{x_k\}$ above the fourth need not be known.

The difficulty of attempting to evaluate (8) directly forces us to attack the problem from a different aspect. Specifically, we adopt a shortcut to obtain, exactly, the variance of whiteness measure W_K. First, observe from (8) that the mean square value of W_K contains a denominator of K^4. Secondly, it has been observed from simulations that the variance of W_K goes to zero proportional to 1/K for large K. Therefore, the form of the variance, V_K, of random variable W_K must be

$$V_K = \mathrm{Var}(W_K) = \frac{A\,K^3 + B\,K^2 + C\,K + D}{K^4}\ , \tag{10}$$

where A, B, C, D are unknown constants. In order to determine these four constants, we will evaluate, exactly, the variance V_K of W_K for a sufficient number of low-order values of K, and then solve the four simultaneous linear equations yielded by (10).

For convenience, we define the sums

$$\phi_n = \sum_{k=n}^{K-1} x_k\, x_{k-n} \quad \text{for } 1 \le n \le K-1\ . \tag{11}$$

Then

$$R_n = \frac{1}{K}\,\phi_n \quad \text{for } 1 \le n \le K-1\ , \tag{12}$$

as seen from (2). The whiteness measure in (3) then takes the form

$$W_K = \frac{2}{K^2}\sum_{n=1}^{K-1} \phi_n^2 \quad \text{for } K \ge 2\ . \tag{13}$$

For K = 1, there are no terms in the sum, yielding $W_1 = 0$.

SPECIAL CASE K = 2

We have, from (11) and (13),

$$\phi_1 = x_1 \, x_0 \; , \quad W_2 = \frac{2}{4} \, \phi_1^2 = \frac{1}{2} \, x_1^2 \, x_0^2 \; . \tag{14}$$

Therefore, upon use of the IID property of the $\{x_k\}$ and (1),

$$E\left(W_2^2\right) = \frac{1}{4} \, E\left(x_1^4 \, x_0^4\right) = \frac{1}{4} \, F^2 \; . \tag{15}$$

The variance of W_2 then follows as

$$V_2 = \mathrm{Var}(W_2) = E\left(W_2^2\right) - E\left(W_2\right)^2 = \frac{1}{4}\left(F^2 - 1\right) \; , \tag{16}$$

where we used (6).

SPECIAL CASE K = 3

The procedure for the remaining cases is similar to that detailed above for K = 2; therefore, the following presentation will be abbreviated, and only the main results will be listed. We have

$$\phi_1 = x_1 \, x_0 + x_2 \, x_1 \; , \quad \phi_2 = x_2 \, x_0 \; , \tag{17}$$

$$W_3 = \frac{2}{9}\left[\phi_1^2 + \phi_2^2\right] = \frac{2}{9}\left[x_1^2 (x_0 + x_2)^2 + x_2^2 \, x_0^2\right] \; , \tag{18}$$

$$W_3^2 = \frac{4}{81}\left[x_1^4 (x_0 + x_2)^4 + x_2^4 \, x_0^4 + 2 \, x_1^2 (x_0 + x_2)^2 \, x_2^2 \, x_0^2\right] \; . \tag{19}$$

The mean value of (19) is given by

$$E\left(W_3^2\right) = \frac{4}{81}\left(F(F + 6 + F) + F^2 + 2(F + F)\right) = \frac{4}{81}\left(3F^2 + 10F\right) . \quad (20)$$

Finally, the variance of W_3 is

$$V_3 = \frac{4}{81}\left(3F^2 + 10F - 9\right) . \quad (21)$$

SPECIAL CASE K = 4

In this case, we have

$$\phi_1 = x_1\ x_0 + x_2\ x_1 + x_3\ x_2\ , \quad \phi_2 = x_2\ x_0 + x_3\ x_1\ , \quad \phi_3 = x_3\ x_0\ , \quad (22)$$

$$W_4 = \frac{2}{16}\left[\phi_1^2 + \phi_2^2 + \phi_3^2\right] , \quad (23)$$

$$64\ W_4^2 = \phi_1^4 + \phi_2^4 + \phi_3^4 + 2\ \phi_1^2\ \phi_2^2 + 2\ \phi_1^2\ \phi_3^2 + 2\ \phi_2^2\ \phi_3^2 . \quad (24)$$

The mean value of (24) will be found in stages. The six components of (24) have the following average values:

$$E\left(\phi_3^4\right) = E\left(x_3^4\ x_0^4\right) = F^2 , \quad (25)$$

$$E\left(\phi_3^2\ \phi_2^2\right) = E\left(x_3^2\ x_0^2(x_2\ x_0 + x_3\ x_1)^2\right) = F + F = 2F , \quad (26)$$

$$E\left(\phi_3^2\ \phi_1^2\right) = E\left(x_3^2\ x_0^2(x_1\ x_0 + x_2\ x_1 + x_3\ x_2)^2\right) = F+1+F = 2F + 1 , \quad (27)$$

$$E\left(\phi_2^4\right) = E\left((x_2\ x_0 + x_3\ x_1)^4\right) = F^2 + 6 + F^2 = 2F^2 + 6 , \quad (28)$$

$$E\left(\phi_2^2\ \phi_1^2\right) = E\left((x_2\ x_0 + x_3\ x_1)^2\ (x_1\ x_0 + x_2\ x_1 + x_3\ x_2)^2\right) =$$

$$= E\left(\left[x_2^2\ x_0^2 + x_3^2\ x_1^2 + 2\ x_3\ x_2\ x_1\ x_0\right]\left[x_1^2\ x_0^2 + x_2^2\ x_1^2 + \right.\right.$$

$$\left.\left. + x_3^2\ x_2^2 + 2\ x_2\ x_1^2\ x_0 + 2\ x_3\ x_2\ x_1\ x_0 + 2\ x_3\ x_2^2\ x_1\right]\right) =$$

$$= F + F + F + F + F + F + 4 = 6F + 4\ , \tag{29}$$

$$\phi_1^2 = x_1^2(x_0 + x_2)^2 + x_3^2\ x_2^2 + 2\ x_3\ x_2\ x_1(x_0 + x_2)\ , \tag{30}$$

$$\phi_1^4 = x_1^4(x_0 + x_2)^4 + x_3^4\ x_2^4 + 6\ x_3^2\ x_2^2\ x_1^2(x_0 + x_2)^2 +$$

$$+ 4\ x_3\ x_2\ x_1^3(x_0 + x_2)^3 + 4\ x_3^3\ x_2^3\ x_1(x_0 + x_2)\ , \tag{31}$$

$$E\left(\phi_1^4\right) = F(F + 6 + F) + F^2 + 6(1 + F) = 3F^2 + 12F + 6\ . \tag{32}$$

Combining these results into (24), we have mean square value

$$E\left(W_4^2\right) = \frac{1}{32}\left(3F^2 + 16F + 11\right) \tag{33}$$

and variance

$$V_4 = \frac{1}{32}\left(3F^2 + 16F - 7\right)\ . \tag{34}$$

The analytical derivations of V_5, V_6, V_7, V_8 are deferred to appendix A due to their lengthy calculations and need for a shorthand notation. It will turn out that we also need all of these latter results when we find the constants A, B, C, D in variance expression (10).

GENERAL DETERMINATION OF VARIANCE OF W_K

The general form for the variance V_K of whiteness measure W_K is given by (10) for arbitrary K and is repeated below:

$$V_K = \mathrm{Var}(W_K) = \frac{A\,K^3 + B\,K^2 + C\,K + D}{K^4} . \tag{35}$$

However, analytic determination of V_K for K = 2, 3, 4, 5, 6, 7, 8 (see appendix A also) have revealed that separate forms like (35) must be employed for K even versus K odd. That is, two <u>different</u> sets of constants A, B, C, D apply in the even versus odd cases of K. The available analytic results for V_K (above and in appendix A) are summarized below:

$$V_1 = 0 \quad \text{(see the line under (13))} , \tag{36}$$

$$V_2 = \frac{1}{4}\left(F^2 - 1\right) , \tag{37}$$

$$V_3 = \frac{4}{81}\left(3F^2 + 10F - 9\right) , \tag{38}$$

$$V_4 = \frac{1}{32}\left(3F^2 + 16F - 7\right) , \tag{39}$$

$$V_5 = \frac{8}{625}\left(5F^2 + 38F - 11\right) , \tag{40}$$

$$V_6 = \frac{1}{324}\left(15F^2 + 144F - 23\right) , \tag{41}$$

$$V_7 = \frac{4}{2401}\left(21F^2 + 246F - 23\right) , \tag{42}$$

$$V_8 = \frac{1}{256}\left(7F^2 + 96F - 3\right) . \tag{43}$$

If we take K equal to the odd values 1, 3, 5, 7 in (35) and use results (36), (38), (40), (42), we obtain four simultaneous linear equations for the constants A, B, C, D. Their solution leads to the following expression for the variance V_K of W_K:

$$V_K = \frac{K-1}{K^4}\left(A\ K^2 + \tilde{B}\ K + \tilde{C}\right) \quad \text{for K odd ,} \tag{44}$$

where

$$A = 4F + \frac{4}{3}\ , \quad \tilde{B} = 2F^2 - 4F - \frac{38}{3}\ , \quad \tilde{C} = -\ 4F + 8\ . \tag{45}$$

When (45) is substituted into (44), the variance expression can be rearranged in terms of powers of F:

$$V_K = \frac{2(K-1)}{K^4}\left[KF^2 + 2\left(K^2-K-1\right)F + \frac{1}{3}\left(2K^2-19K+12\right)\right] \quad \text{for K odd .} \tag{46}$$

The F^2 term here confirms (9), as anticipated.

If we take K equal to the even values 2, 4, 6, 8 in (35) and use results (37), (39), (41), (43), we obtain four different simultaneous linear equations for the constants A, B, C, D. Their solution leads to the following expression for the variance V_K of W_K:

$$V_K = \frac{1}{K^3}\left(A\ K^2 + B\ K + C\right) \quad \text{for K even ,} \tag{47}$$

where

$$A = 4F + \frac{4}{3}\ , \quad B = 2F^2 - 8F - 14\ , \quad C = -\ 2F^2 + \frac{62}{3}\ . \tag{48}$$

When (48) is substituted into (47), the variance expression can be rearranged in terms of powers of F according to

$$V_K = \frac{2}{K^3}\left[(K-1)F^2 + 2K(K-2)F + \frac{1}{3}\left(2K^2-21K+31\right)\right] \quad \text{for K even .} \tag{49}$$

Again, the F^2 dependence in (9) is confirmed.

The asymptotic behavior of variance V_K for large K is given by

$$V_K \sim \left(4F + \frac{4}{3}\right)\frac{1}{K} \quad \text{as } K \to \infty \tag{50}$$

for both K odd and K even. This is due to the fact that constant A in (35) is identical for the odd and even cases; compare (45) and (48). Thus, whiteness measure W_K tends to cluster around 1 as $K \to \infty$. Recall that $R_0 \to 1$, while $R_n \to 0$ for fixed n, as $K \to \infty$.

The end results for variance V_K of whiteness measure W_K are given by (44) and (47), or by (46) and (49). Plots of V_K for the uniform random variable and the Gaussian random variable $\{x_k\}$ are displayed in figures 1 and 2, respectively. A short tabulation of V_K is given in table 1 for the uniform, Gaussian, exponential, and alternating random variables $\{x_k\}$. The probability density functions of $\{x_k\}$ for these four cases are, respectively,

$$p_u(x) = .5/\sqrt{3} \quad \text{for } |x| < \sqrt{3} \ , \quad F = 1.8 \ ; \tag{51}$$

$$p_g(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) \ , \quad F = 3 \ ; \tag{52}$$

$$p_e(x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|) \ , \quad F = 6 \ ; \tag{53}$$

$$p_a(x) = \frac{1}{2}\,\delta(x-1) + \frac{1}{2}\,\delta(x+1) \ , \quad F = 1 \ . \tag{54}$$

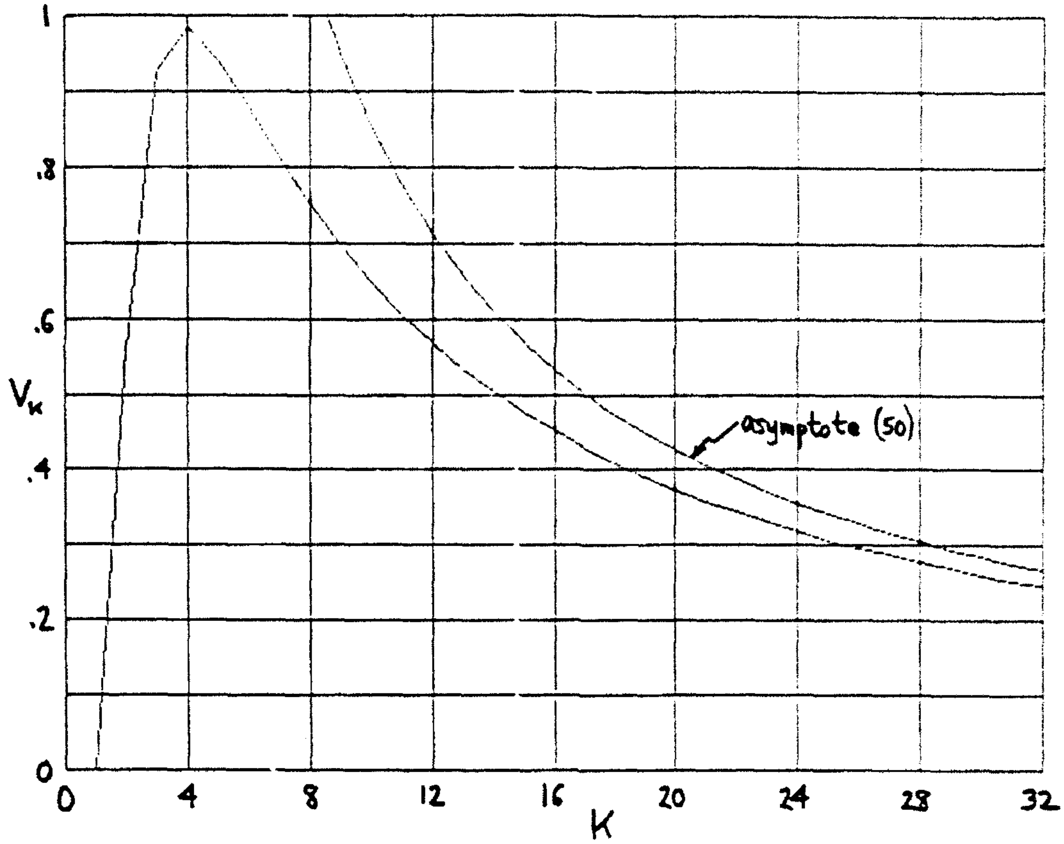

Figure 1. Variance V_K for Uniform Random Variables $\{x_K\}$

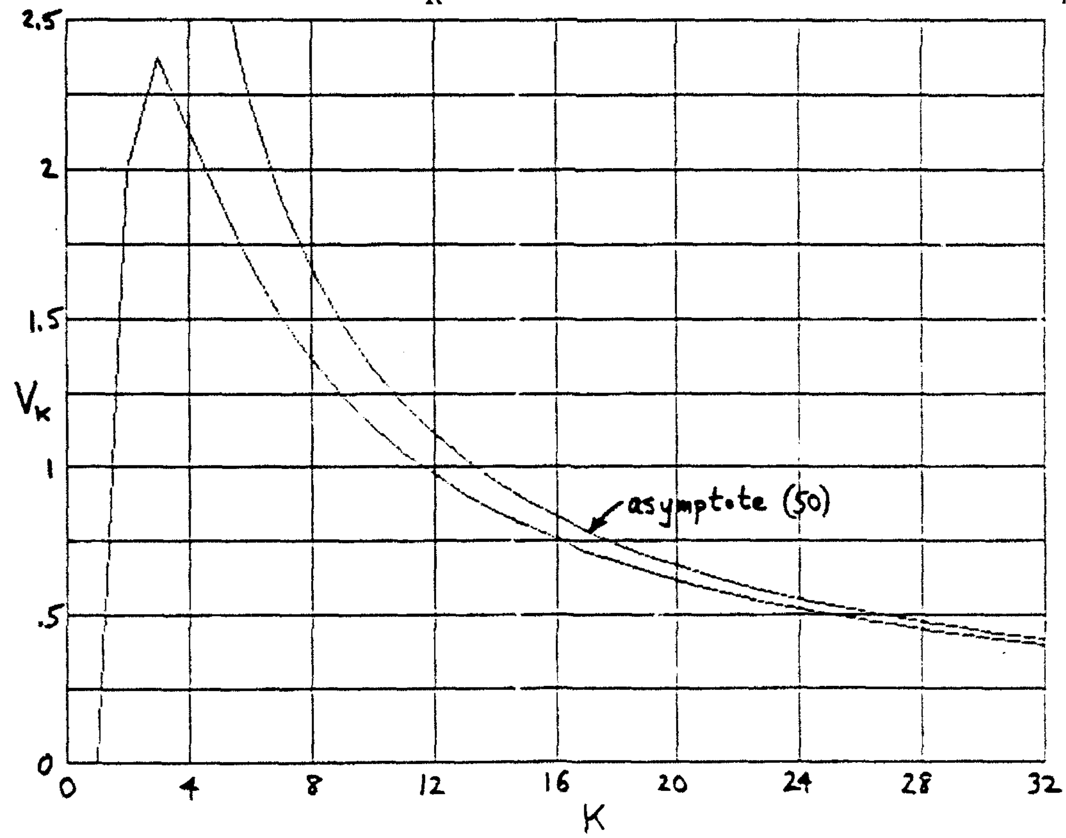

Figure 2. Variance V_K for Gaussian Random Variables $\{x_K\}$

A short table of the variances for these four examples is given below. For the alternating example, $x_k = \pm 1$ and $F = 1$, whiteness measure W_2 for $K = 2$ is always equal to 1/2, thereby leading to variance $V_2 = 0$. The smallest possible example of F is 1, as realized in the alternating random variable case.

Table 1. Variance V_K of Whiteness Measure W_K

K	Uniform	Gaussian	Exponential	Alternating
2	.56	2.	8.75	.0
3	.92444	2.37037	7.85185	.19753
4	.985	2.125	6.15625	.375
5	.94208	1.8944	5.0816	.4096
6	.87901	1.67901	4.26235	.41975
7	.81273	1.50604	3.68013	.40650
8	.75188	1.35938	3.22266	.39063
16	.45117	.75586	1.60986	.25977
32	.24569	.39722	.79987	.14771
64	.12804	.20346	.39808	.07852
128	.06534	.10295	.19850	.04045

If we combine (47) with the multiplied-out version of (44), the variance V_K can indeed be written in the form (35) for <u>all</u> K, where the constants A, B, C are as given in (48), but constant D must be taken according to the two different values

$$D = \begin{Bmatrix} 0 & \text{for } K \text{ even} \\ 4(F-2) & \text{for } K \text{ odd} \end{Bmatrix} . \tag{55}$$

Notice that, despite (8) involving eighth-order products, nothing above fourth-order moment F of $\{x_k\}$ is required in these results.

PROBABILITY DISTRIBUTIONS OF WHITENESS MEASURE

The direct evaluation of whiteness measure W_K, according to its definition (3) in conjunction with (2), is very time consuming for large K, due to all the multiplications required. An attractive alternative, in terms of fast Fourier transforms, was derived in [1; appendix C] and is employed here; the program utilized is listed in appendix B. The key relation relative to (3) is [1; (C-5)]

$$W_K = \frac{1}{K^2 M^2}\left[M \sum_{m=0}^{M-1} |X_m|^4 - \left(\sum_{m=0}^{M-1} |X_m|^2\right)^2\right] , \tag{56}$$

where M is the size of the fast Fourier transform $\{X_m\}$ of data $\{x_k\}$. The only restriction on M is that we must use $M \geq 2K - 1$; then, the right-hand side of (56) is independent of M. (For $K = 1$, $X_m = x_0$ for $0 \leq m \leq M-1$, leading to $W_1 = 0$, as noted under (13).) Again, notice that the whiteness measure W_K depends on fourth-order products of the data or its transform.

The cumulative distribution function (CDF) and exceedance distribution function (EDF) of whiteness measure W_K,

$$\text{CDF}(u) = \text{Prob}(W_K < u) , \quad \text{EDF}(u) = \text{Prob}(W_K > u) , \tag{57}$$

for the case where data $\{x_k\}$ is uniformly distributed over $-\sqrt{3},\sqrt{3}$ [see (51)], are displayed in figures 3 - 10 for K = 2, 3, 4, 8, 16, 32, 64, 128, respectively. These results were determined by using at least one million trials for W_K as defined in (56). The

exceedance distribution function for small K has a cusp near zero argument which disappears for larger K. However, random variable W_K does not approach Gaussian as K increases; rather, as shown in figure 10 for K = 128, the right-hand tail appears to approach exponential behavior. For a bounded random variable, $|x_k| < B$, the value of W_K is bounded according to

$$W_K < \frac{(K-1)(2K-1)}{3K} B^4 . \tag{58}$$

In the case of the uniform random variable x_k, where $B = \sqrt{3}$, (58) yields 4.5 for K = 2, 10 for K = 3, and 15.75 for K = 4.

Although the mean of W_{128} is 127/128 and its variance is $V_{128} = .06534$, the standard deviation of W_{128} is 0.256; this leads to the possibility of large values of W_{128} on occasion. For example, figure 10 shows that the whiteness measure can reach a value of 1.8 or larger about 1% of the time. If a candidate uniform random number generator has probability distributions for W_K which differ significantly from figures 3 - 10, it is suspect and should be more thoroughly investigated before further use.

The corresponding cumulative and exceedance distribution functions of the whiteness measure W_K for a Gaussian random number generator [see (52)] are displayed in figures 11 - 18 for K = 2, 3, 4, 8, 16, 32, 64, 128, respectively. The first observation to make is that the positive tail of W_K can now reach much larger values when K is small. However, for the larger values of K, the probability distributions of W_K appear to be approaching a common behavior, regardless of the distribution of the underlying data $\{x_k\}$; compare figures 10 and 18 for K = 128.

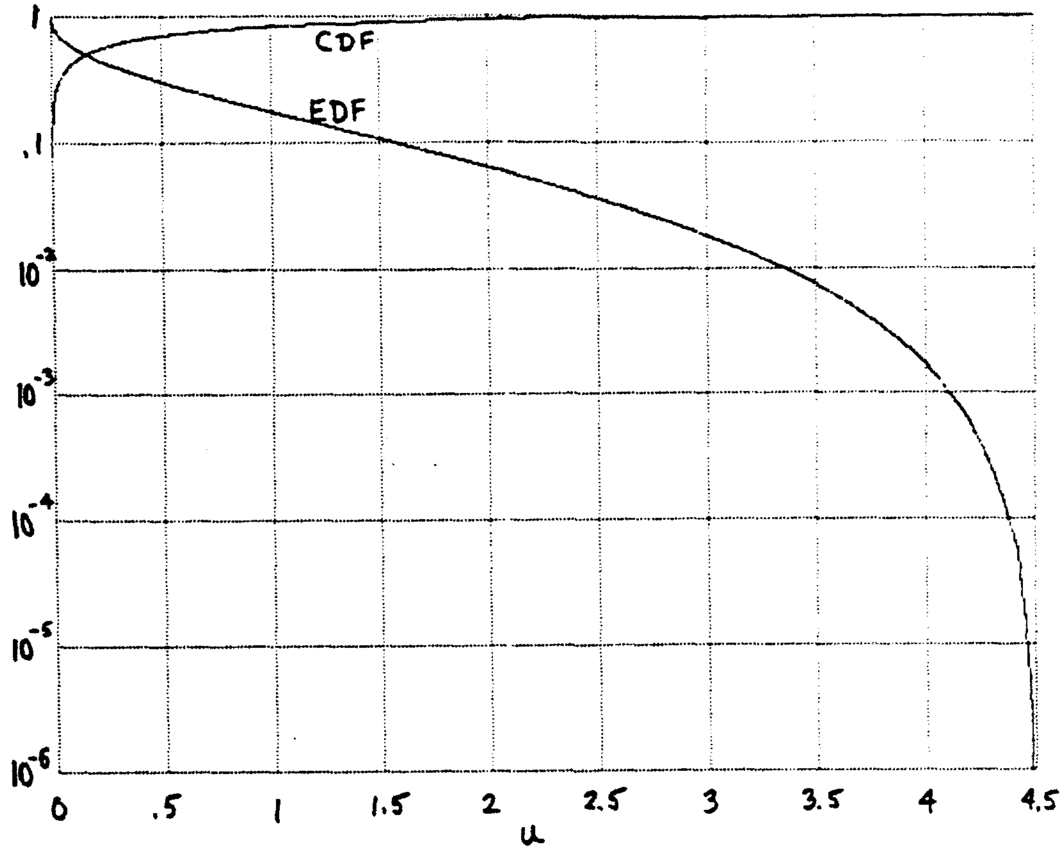

Figure 3. Distributions of W_2 for Uniform Random Variables

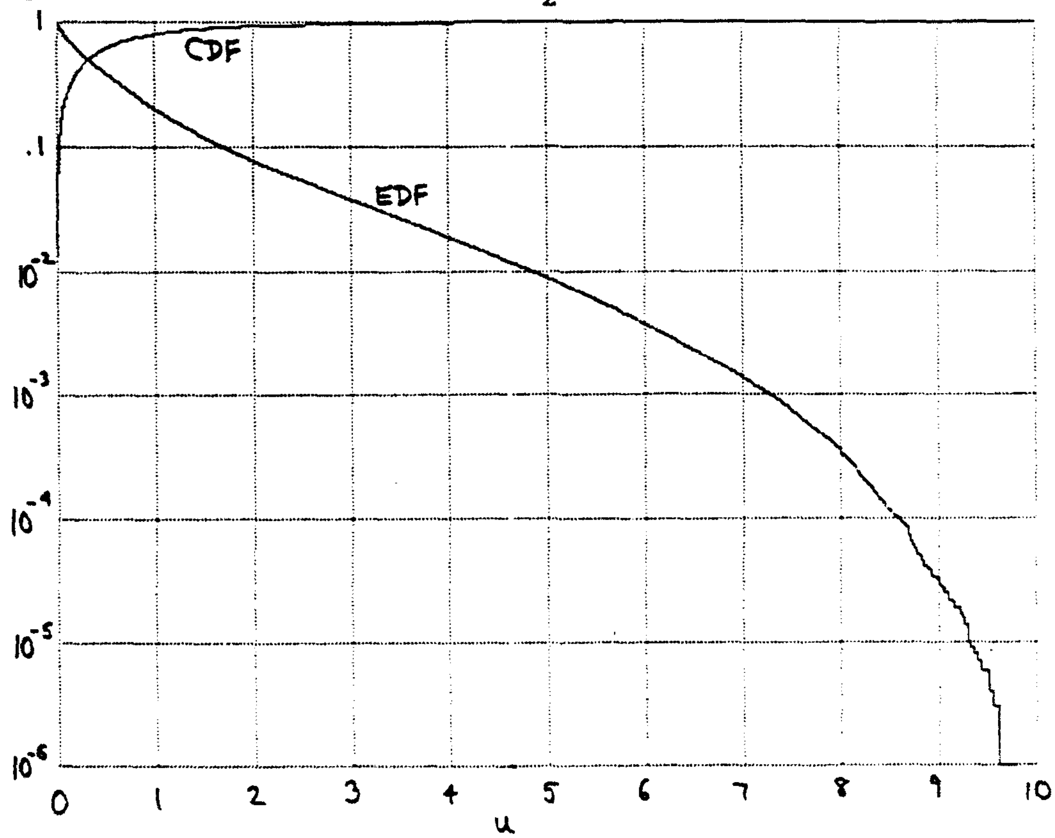

Figure 4. Distributions of W_3 for Uniform Random Variables

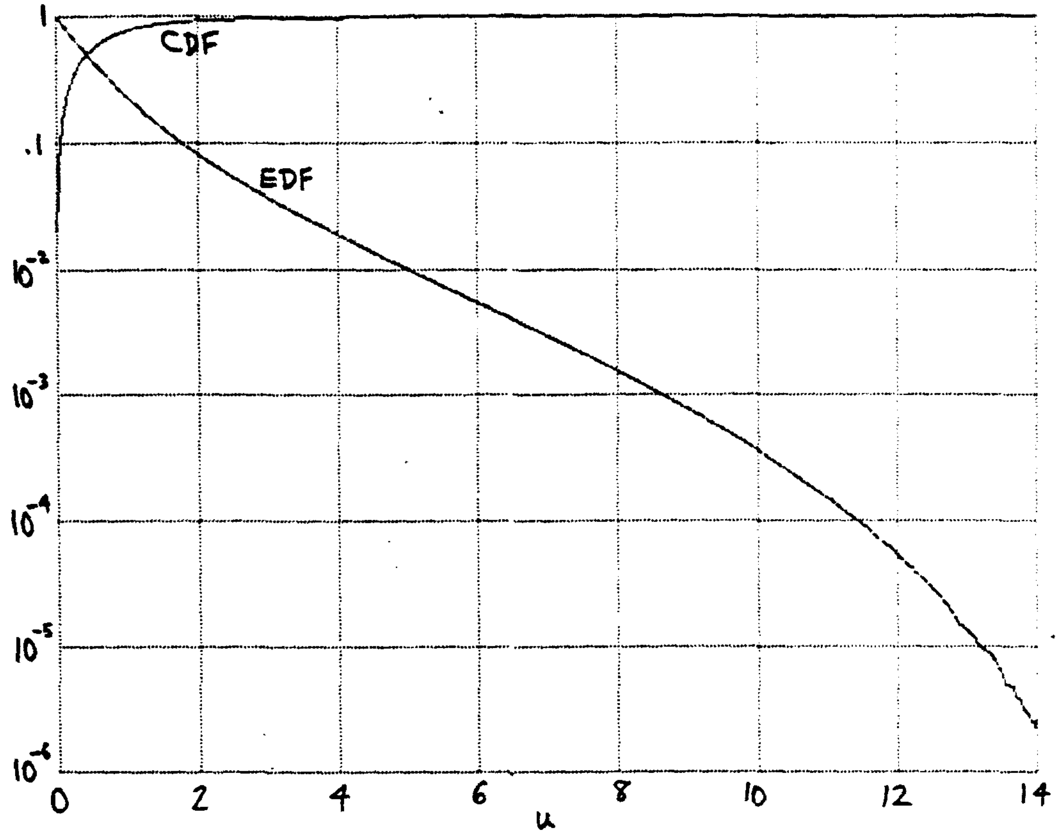

Figure 5. Distributions of W_4 for Uniform Random Variables

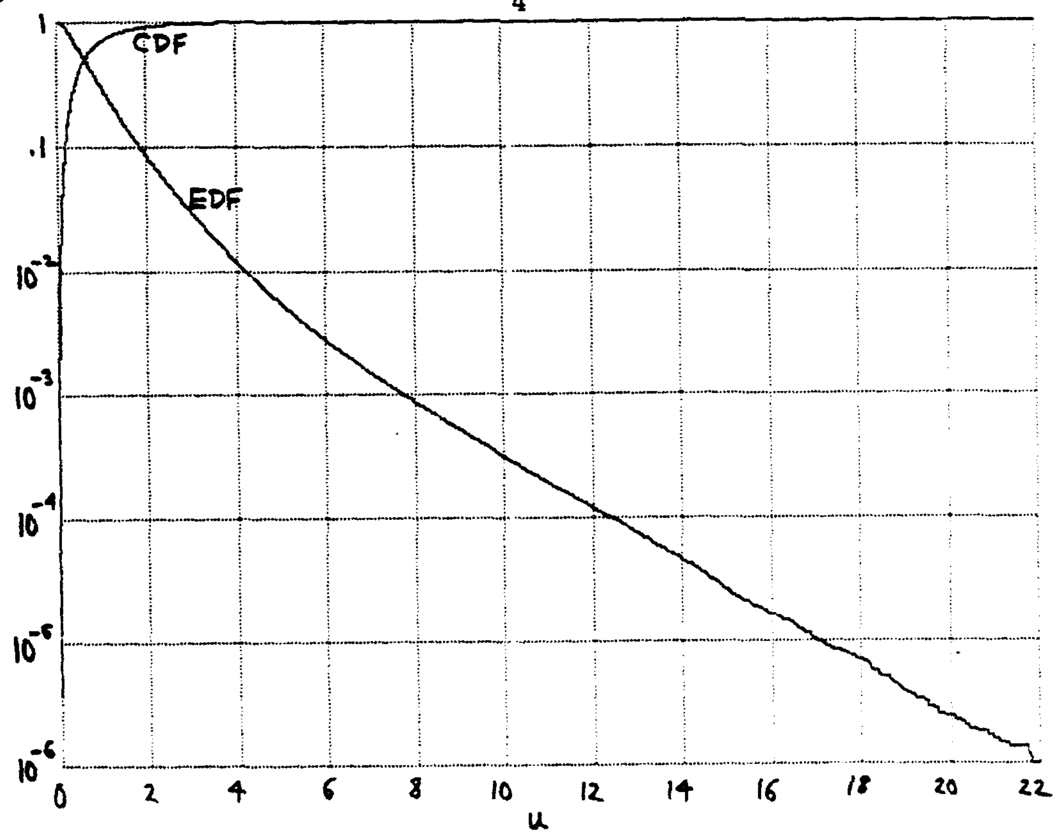

Figure 6. Distributions of W_8 for Uniform Random Variables

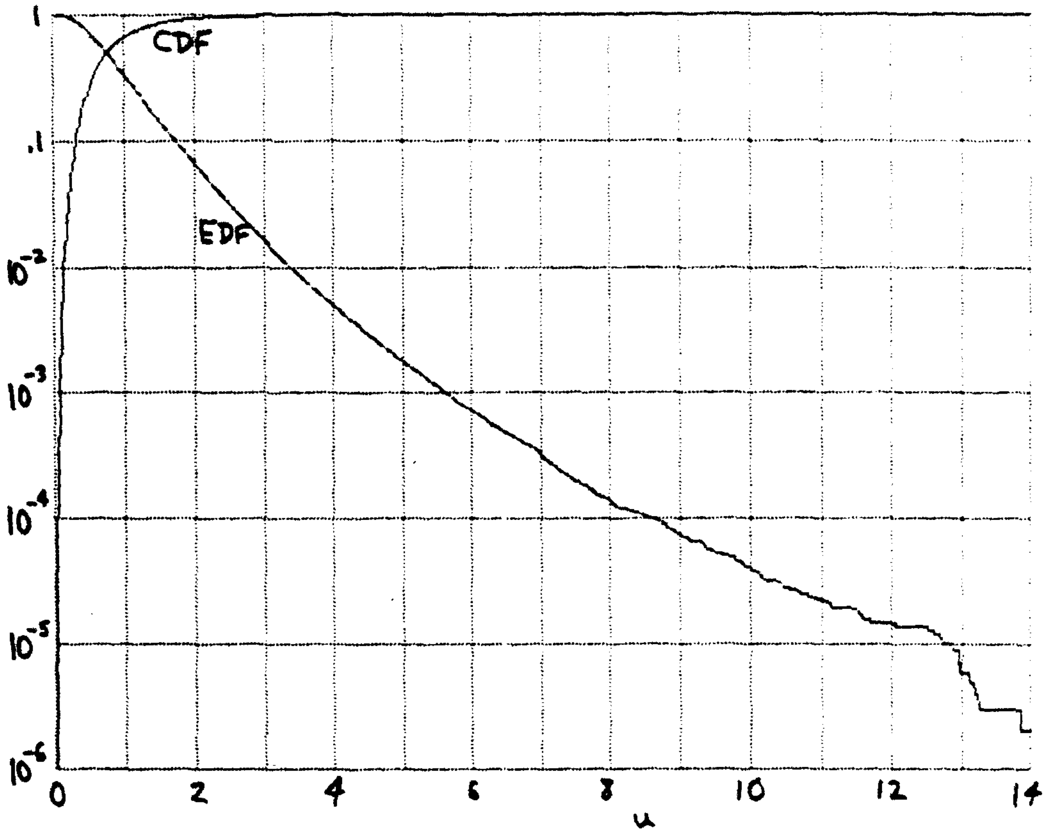

Figure 7. Distributions of W_{16} for Uniform Random Variables

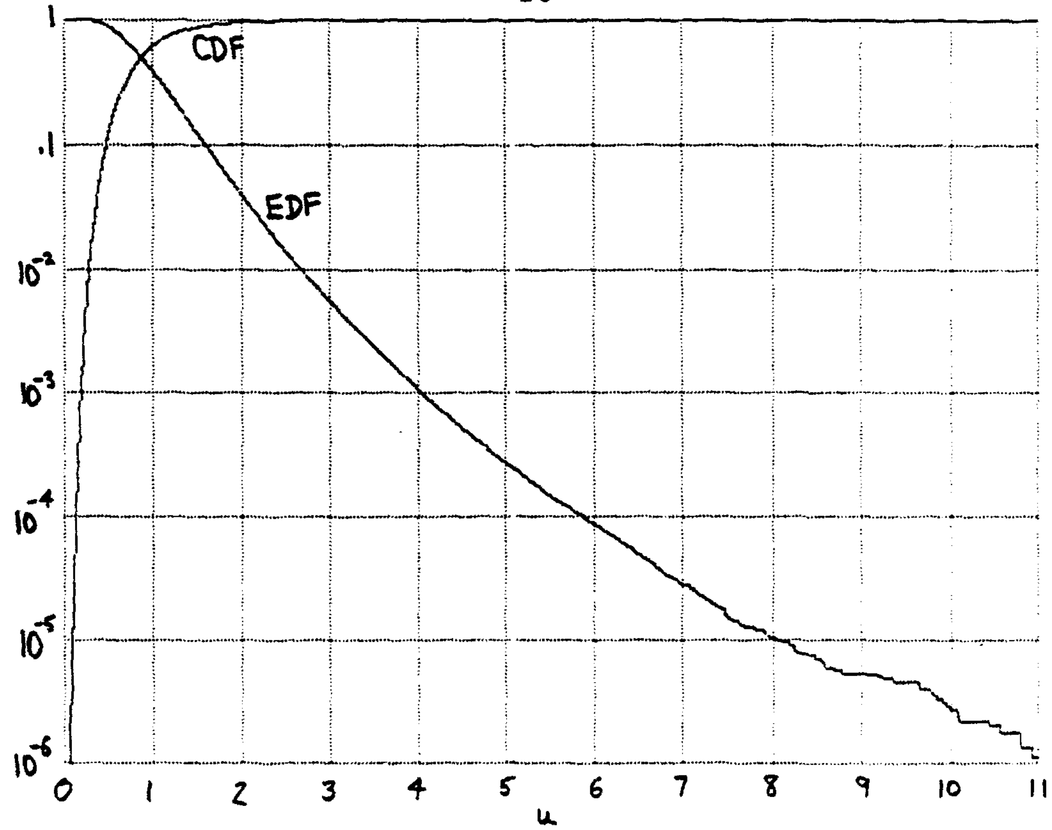

Figure 8. Distributions of W_{32} for Uniform Random Variables

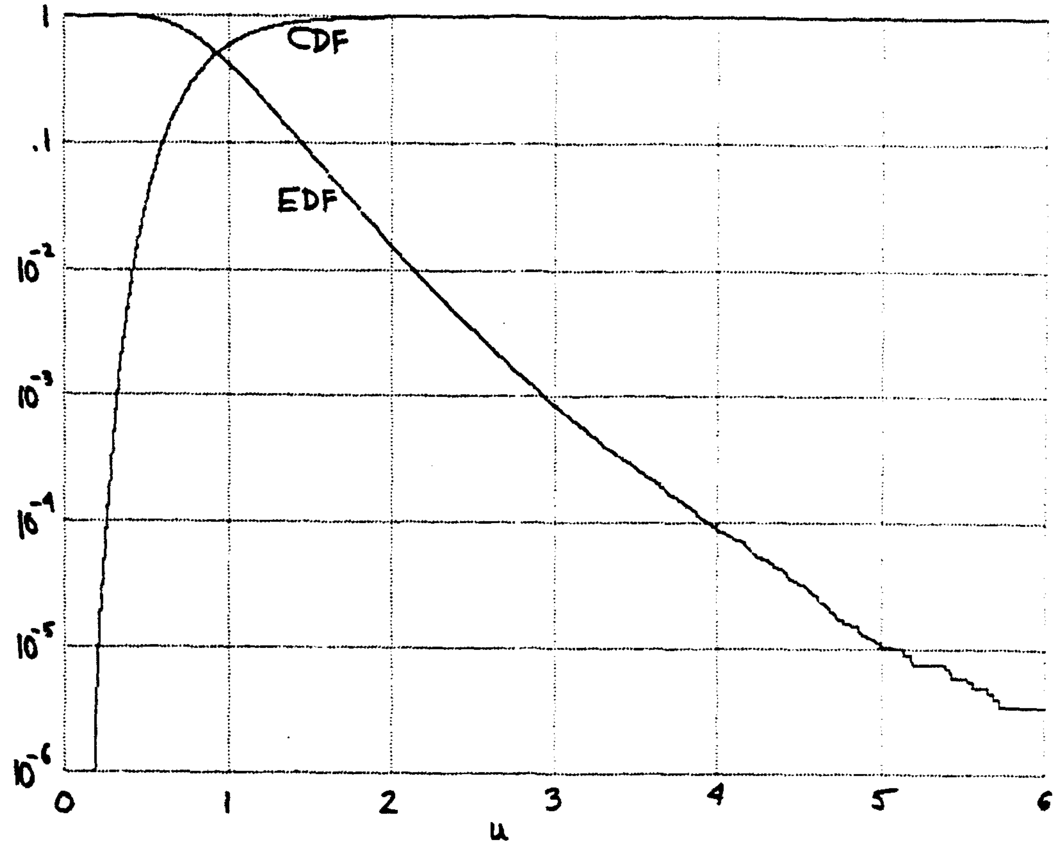

Figure 9. Distributions of W_{64} for Uniform Random Variables

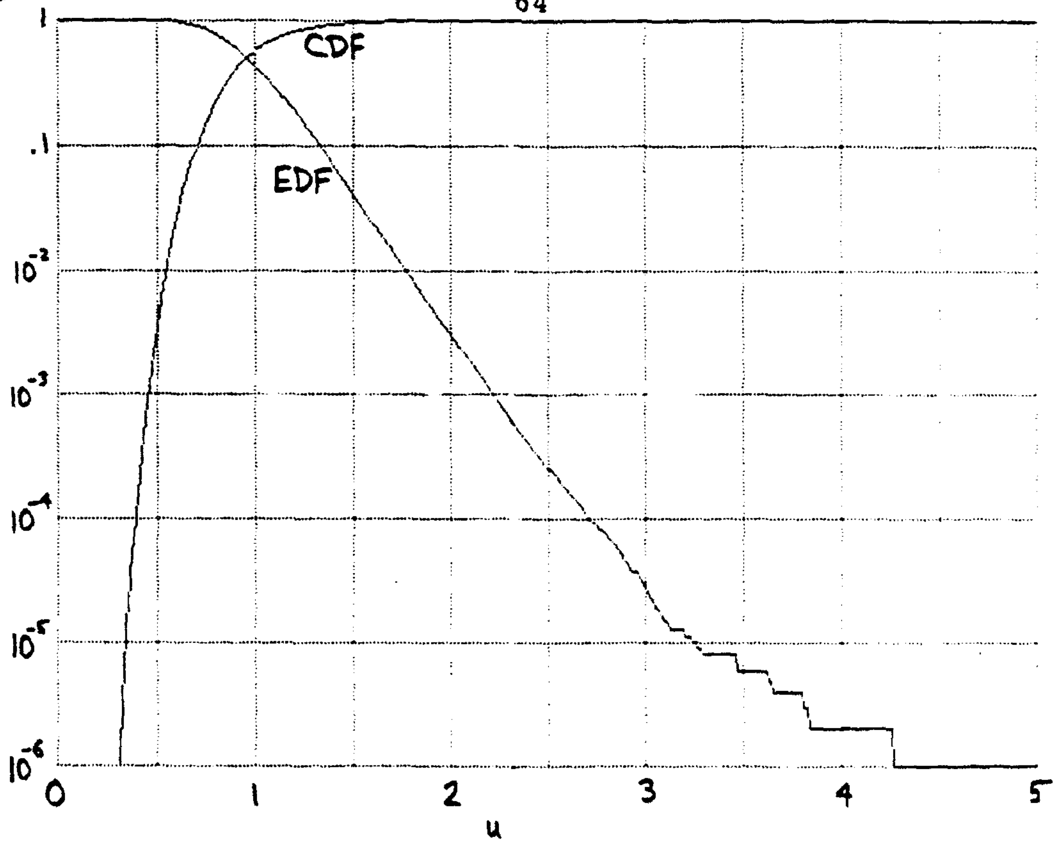

Figure 10. Distributions of W_{128} for Uniform Random Variables

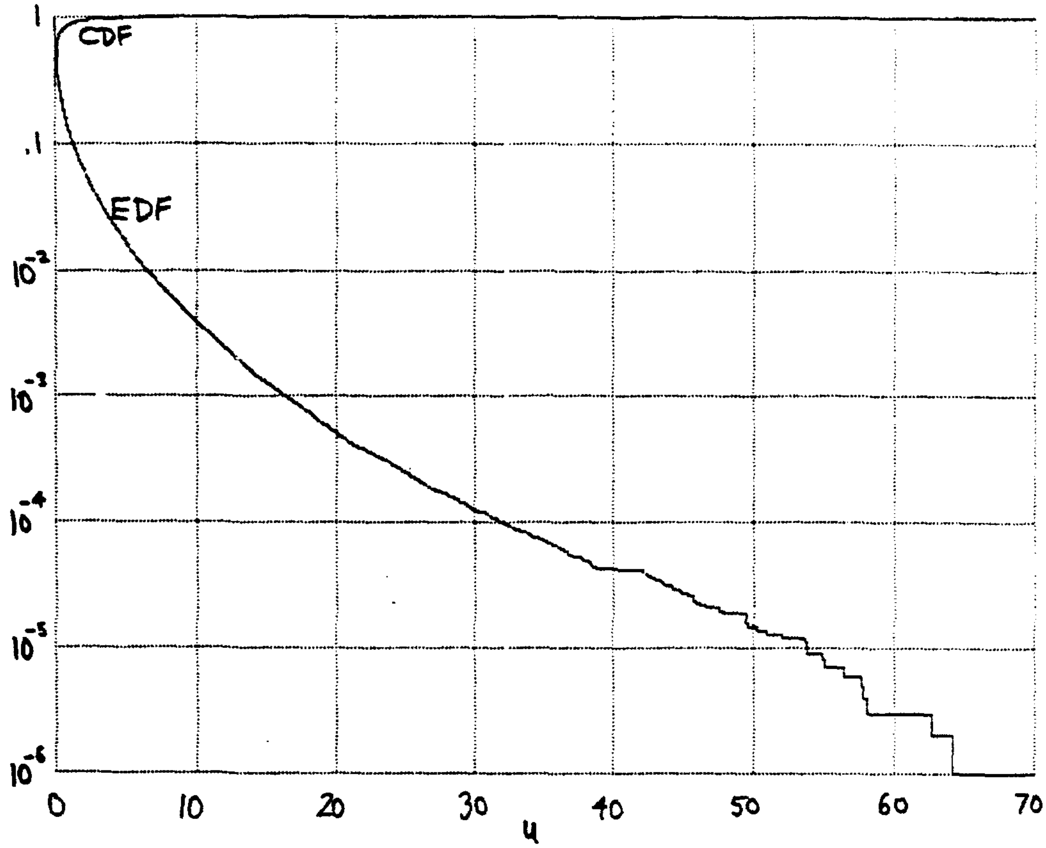

Figure 11. Distributions of W_2 for Gaussian Random Variables

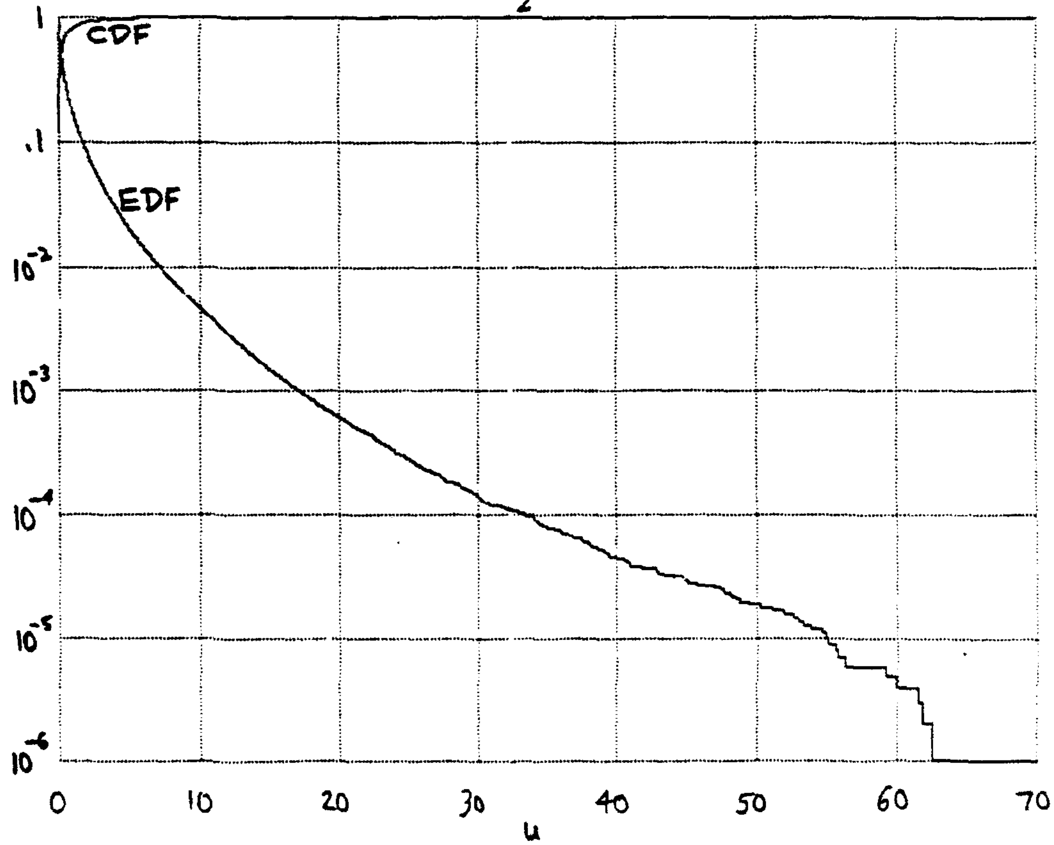

Figure 12. Distributions of W_3 for Gaussian Random Variables

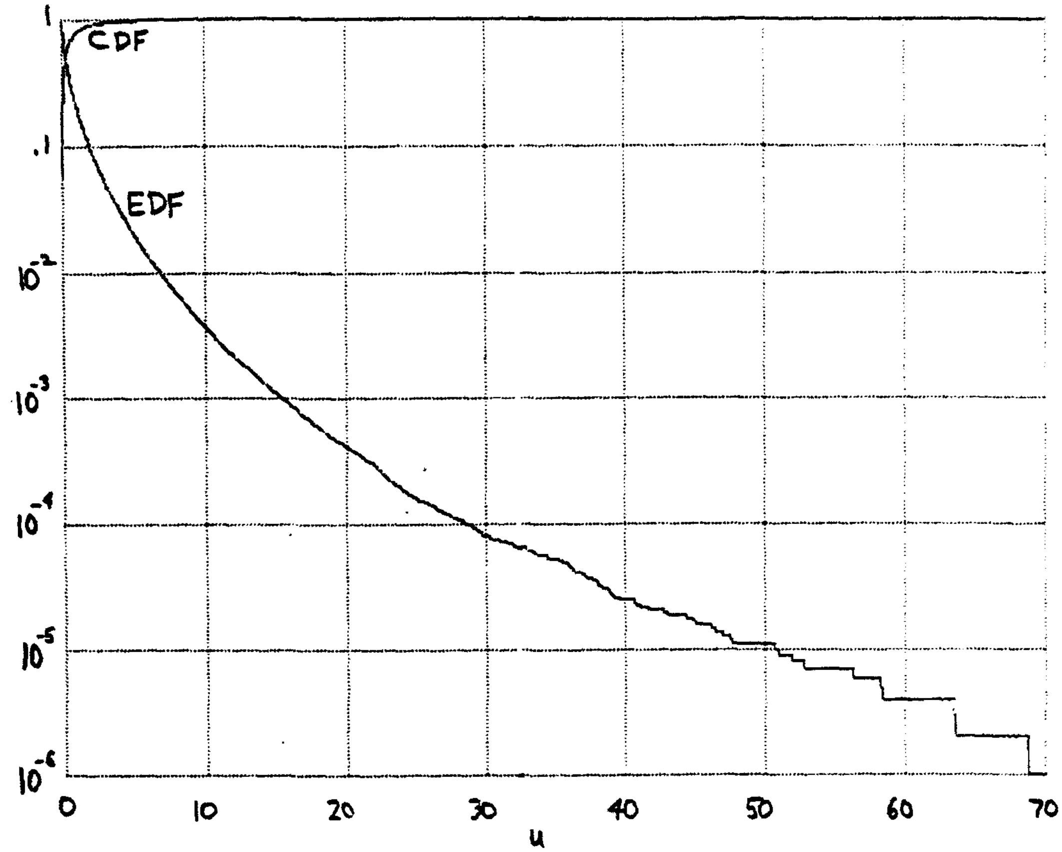

Figure 13. Distributions of W_4 for Gaussian Random Variables

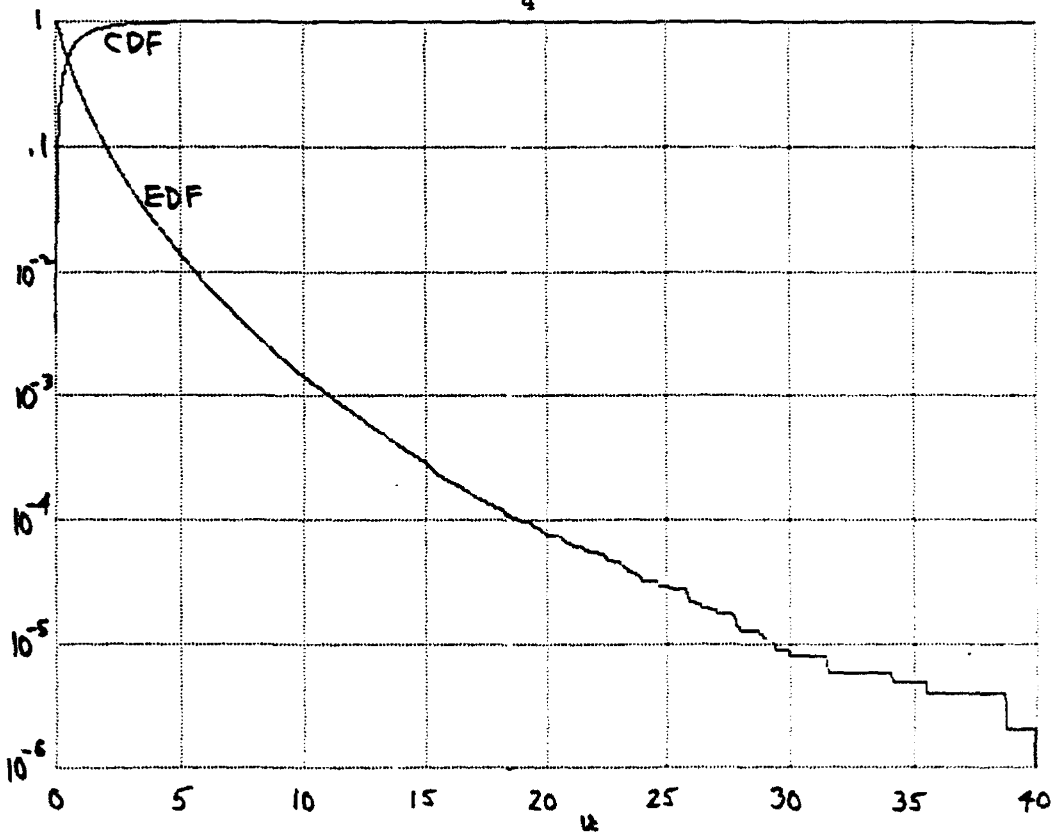

Figure 14. Distributions of W_8 for Gaussian Random Variables

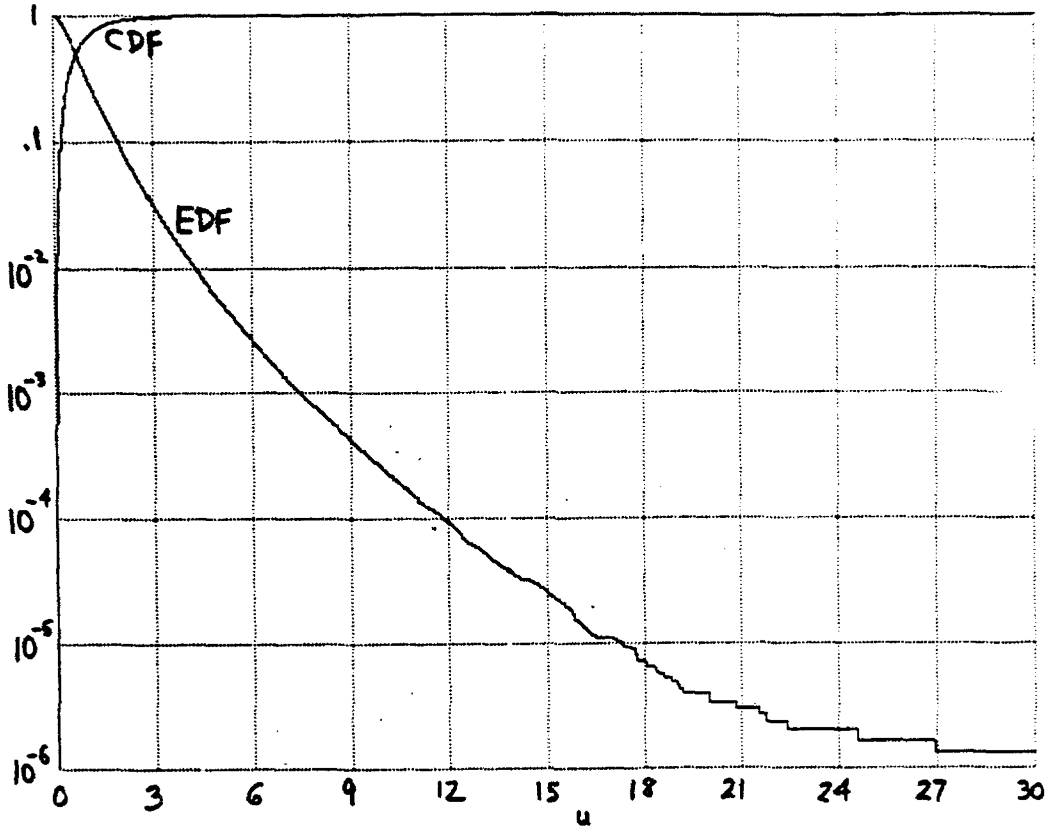

Figure 15. Distributions of W_{16} for Gaussian Random Variables

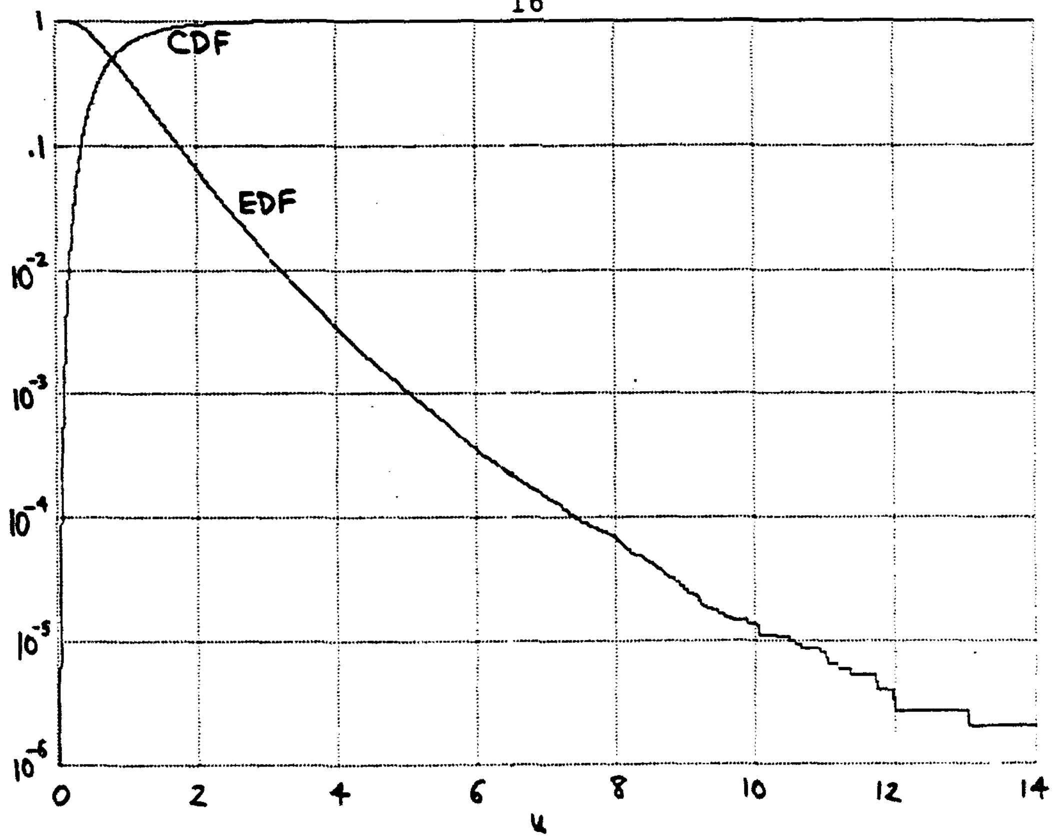

Figure 16. Distributions of W_{32} for Gaussian Random Variables

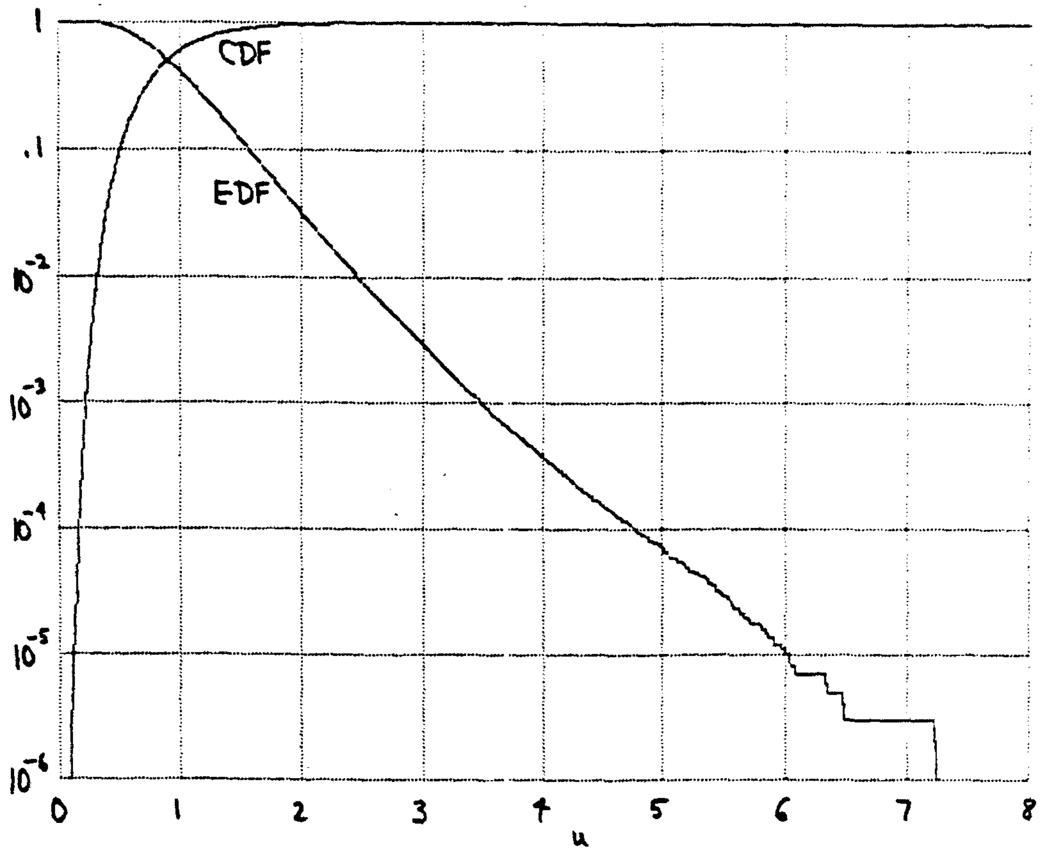

Figure 17. Distributions of W_{64} for Gaussian Random Variables

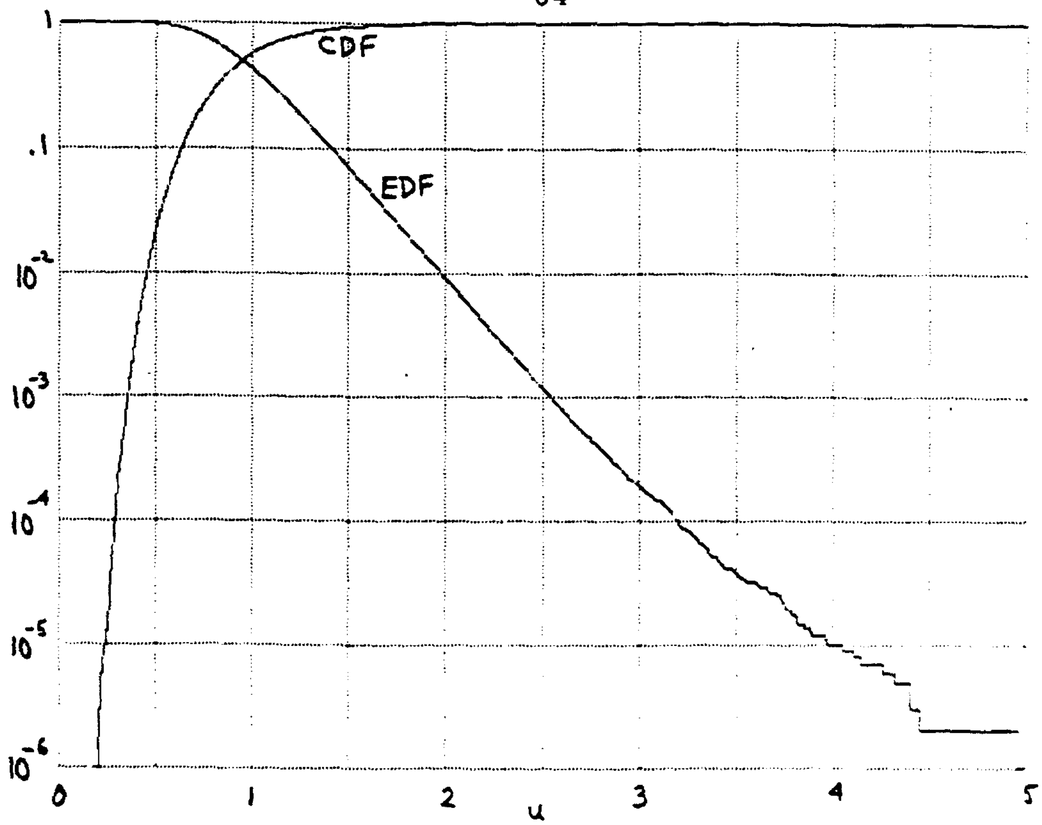

Figure 18. Distributions of W_{128} for Gaussian Random Variables

SUMMARY

The statistics of a whiteness measure, for testing a random number generator, have been investigated in terms of the mean, variance, and probability distributions. The mean and variance results are exact and have been borne out by numerous simulations for different noise sources $\{x_k\}$ and data sizes K. These results, for whiteness measure W_K defined in (3), are summarized below:

$$E(W_K) = \frac{K - 1}{K} , \quad V_K = \mathrm{Var}(W_K) = \frac{A\,K^3 + B\,K^2 + C\,K + D}{K^4} , \tag{59}$$

where

$$A = 4F + \frac{4}{3} , \quad B = 2F^2 - 8F - 14 , \quad C = -\,2F^2 + \frac{62}{3} \quad \text{for all } K ,$$

$$\text{while} \quad D = \begin{Bmatrix} 0 & \text{for } K \text{ even} \\ 4(F - 2) & \text{for } K \text{ odd} \end{Bmatrix} . \tag{60}$$

The mean of whiteness measure W_K is independent of fourth-order moment F, while the variance of W_K depends on F, but not on sixth or eighth-order moments of data $\{x_k\}$. That is, the eighth-order product encountered in the general mean-square expression (8) never requires knowledge higher than fourth-order for its evaluation. This result applies for a symmetric zero-mean probability density function for unit-variance data $\{x_k\}$.

The cumulative and exceedance probability distributions were determined by simulations involving more than one million trials each and therefore have good reliability approximately down to the .0001 probability level.

APPENDIX A. DERIVATION OF VARIANCE OF WHITENESS MEASURE W_K

The variances V_K of whiteness measure W_K for K = 2, 3, 4 were derived in (14) - (34) in the main text. We now present the derivations for the remaining cases, K = 5, 6, 7, 8, that are necessary in order to determine V_K for all K.

SPECIAL CASE K = 5

$$\phi_1 = x_1 x_0 + x_2 x_1 + x_3 x_2 + x_4 x_3 , \qquad \phi_4 = x_4 x_0 ,$$

$$\phi_2 = x_2 x_0 + x_3 x_1 + x_4 x_2 , \quad \phi_3 = x_3 x_0 + x_4 x_1 , \tag{A-1}$$

$$W_5 = \frac{2}{25}\left[\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right] , \tag{A-2}$$

$$\frac{625}{4} W_5^2 = \phi_1^4 + \phi_2^4 + \phi_3^4 + \phi_4^4 + 2 \phi_1^2 \phi_2^2 + 2 \phi_1^2 \phi_3^2 + 2 \phi_1^2 \phi_4^2 +$$

$$+ 2 \phi_2^2 \phi_3^2 + 2 \phi_2^2 \phi_4^2 + 2 \phi_3^2 \phi_4^2 . \tag{A-3}$$

The component averages required are developed in detail as follows:

$$E\left(\phi_4^4\right) = E\left(x_4^4 x_0^4\right) = F^2 , \tag{A-4}$$

$$E\left(\phi_4^2 \phi_3^2\right) = E\left(x_4^2 x_0^2 (x_3 x_0 + x_4 x_1)^2\right) = F + F = 2F , \tag{A-5}$$

$$E\left(\phi_4^2 \phi_2^2\right) = E\left(x_4^2 x_0^2 [x_3 x_1 + x_2 (x_0 + x_4)]^2\right) =$$

$$= E\left(x_4^2 x_0^2 \left[x_3^2 x_1^2 + x_2^2 (x_0 + x_4)^2 + 2 x_3 x_2 x_1 (x_0 + x_4)\right]\right) =$$

$$= 1 + (F + F) = 2F + 1 \,, \tag{A-6}$$

$$E\left(\phi_4^2\ \phi_1^2\right) = E\left(x_4^2\ x_0^2[x_1(x_0 + x_2) + x_3(x_2 + x_4)]^2\right) =$$

$$= E\left(x_4^2\ x_0^2\left[x_1^2(x_0+x_2)^2 + x_3^2(x_2+x_4)^2 + 2\ x_3\ x_1(x_0+x_2)(x_2+x_4)\right]\right) =$$

$$= (F + 1) + (1 + F) = 2F + 2 \,, \tag{A-7}$$

$$E\left(\phi_3^4\right) = E\left([x_3\ x_0 + x_4\ x_1]^4\right) = F^2 + 6 + F^2 = 2F^2 + 6 \,, \tag{A-8}$$

$$E\left(\phi_3^2\ \phi_2^2\right) = E\left([x_3\ x_0 + x_4\ x_1]^2\ [x_3\ x_1 + x_2(x_4 + x_0)]^2\right) =$$

$$= E\left(\left[x_3^2\ x_0^2 + x_4^2\ x_1^2 + 2\ x_4\ x_3\ x_1\ x_0\right]\left[x_3^2\ x_1^2 + x_2^2(x_4 + x_0)^2 +\right.\right.$$

$$\left.\left.+ 2\ x_3\ x_2\ x_1(x_4 + x_0)\right]\right) = F + (1+F) + F + (F+1) = 4F + 2 \,, \tag{A-9}$$

$$E\left(\phi_3^2\ \phi_1^2\right) = E\left([x_3\ x_0 + x_4\ x_1]^2\ [x_1(x_2 + x_0) + x_3(x_4 + x_2)]^2\right) =$$

$$= E\left(\left[x_3^2\ x_0^2 + x_4^2\ x_1^2 + 2\ x_4\ x_3\ x_1\ x_0\right]\left[x_1^2(x_2 + x_0)^2 + x_3^2(x_4 + x_2)^2 +\right.\right.$$

$$\left.\left.+ 2\ x_3\ x_1(x_2 + x_0)(x_4 + x_2)\right]\right) =$$

$$= (1 + F) + F(1 + 1) + F(1 + 1) + (F + 1) + 4 = 6F + 6 \,, \tag{A-10}$$

$$E\left(\phi_2^4\right) = E\left([x_3\ x_1 + x_2(x_4 + x_0)]^4\right) =$$

$$= F^2 + 6(1 + 1) + F(F + 6 + F) = 3F^2 + 6F + 12, \tag{A-11}$$

$$E\left(\phi_2^2\ \phi_1^2\right) = E\left([x_3\ x_1 + x_2(x_4 + x_0)]^2\ [x_1(x_2 + x_0) + x_3(x_4 + x_2)]^2\right)$$

$$= E\left(\left[x_3^2\ x_1^2 + x_2^2(x_4 + x_0)^2 + 2\ x_3\ x_2\ x_1(x_4 + x_0)\right] \times\right.$$

$$\left.\times\ \left[x_1^2(x_2 + x_0)^2 + x_3^2(x_4 + x_2)^2 + 2\ x_3\ x_1(x_2 + x_0)(x_4 + x_2)\right]\right) =$$

$$= F(1 + 1) + F(1 + 1) +$$

$$+ E\left(\left[x_4^2 + 2\ x_4\ x_0 + x_0^2\right]\left[x_2^4 + 2\ x_2^3\ x_0 + x_2^2\ x_0^2\right]\right) +$$

$$+ E\left(\left[x_4^2 + 2\ x_4\ x_0 + x_0^2\right]\left[x_4^2\ x_2^2 + 2\ x_4\ x_2^3 + x_2^4\right]\right) +$$

$$+ 4\ E\left(x_2(x_4 + x_0)(x_2 + x_0)(x_4 + x_2)\right) =$$

$$= 4F + (F + 1 + F + F) + (F + F + 1 + F) + 4(1 + 1) = 10F + 10, \tag{A-12}$$

$$E\left(\phi_1^4\right) = E\left([x_1(x_2 + x_0) + x_3(x_4 + x_2)]^4\right) =$$

$$= E\left(x_1^4(x_2 + x_0)^4 + 6\ x_1^2(x_2 + x_0)^2\ x_3^2(x_4 + x_2)^2 + x_3^4(x_4 + x_2)^4\right) =$$

$$= F(F + 6 + F) + 6\ E\left(\left[x_2^2 + 2\ x_2\ x_0 + x_0^2\right]\left[x_4^2 + 2\ x_4\ x_2 + x_2^2\right]\right) +$$

$$+ F(F + 6 + F) = 4F^2 + 12F + 6(1 + F + 1 + 1) =$$

$$= 4F^2 + 18F + 18\ . \tag{A-13}$$

Now, we combine all the component averages, above, to obtain mean square value

$$E\left(W_5^2\right) = \frac{8}{625}(5F^2 + 38F + 39) \qquad \text{(A-14)}$$

and variance

$$V_5 = \frac{8}{625}(5F^2 + 38F - 11) \ . \qquad \text{(A-15)}$$

SPECIAL CASE K = 6

Now, we adopt a very useful shorthand notation to handle the rest of the cases of interest. For example, here, $\phi_5 = x_5\ x_0$ and $\phi_5^2 = x_5^2\ x_0^2$, which is denoted by 5500; that is, the superfluous x is ignored when possible. Also, $x_4\ x_2^2\ x_0$ is denoted by 4220. With this background, we now have

$$\phi_5^2 = 5500 \ , \quad \phi_4^2 = 4400+5511+2(5410) \ ,$$

$$\phi_3^2 = 3300+4411+5522+2(4310+5320+5421) \ ,$$

$$\phi_2^2 = 2200+3311+4422+5533+2(3210+4220+5320+4321+5331+5432) \ ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+$$

$$+2(2110+3210+4310+5410+3221+4321+5421+4332+5432+5443) \ . \qquad \text{(A-16)}$$

From (13), there follows

$$W_6 = \frac{2}{36}\sum_{n=1}^{5} \phi_n^2 = \frac{1}{18}\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_5^2\right) \qquad \text{(A-17)}$$

and

$$324\ W_6^2 = \phi_1^4+\phi_2^4+\phi_3^4+\phi_4^4+\phi_5^4 + 2\left(\phi_1^2\phi_2^2 + \cdots + \phi_4^2\phi_5^2\right) . \qquad \text{(A-18)}$$

We also abbreviate the following ensemble averages as follows

$$E\left(\phi_m^2\ \phi_n^2\right) = Tmn . \qquad \text{(A-19)}$$

Then, there follows, in a straightforward but tedious manner,

$$T55 = F^2 ,\quad T54 = F+F = 2F ,\quad T53 = F+1+F = 2F+1 ,$$

$$T52 = F+1+1+F = 2F+2 ,\quad T51 = F+1+1+1+F = 2F+3 ,$$

$$T44 = F^2+6+F^2 = 2F^2+6 ,\quad T43 = (F+F+1)+(1+F+F) = 4F+2 ,$$

$$T42 = (F+1+F+1)+(1+F+1+F) = 4F+4 ,$$

$$T41 = (F+1+1+F+F)+(F+F+1+1+F)+4 = 6F+8 ,$$

$$T33 = 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 ,$$

$$T32 = (F+F+1+F)+(1+F+F+1)+(F+1+F+F)+4(1) = 8F+8 ,$$

$$T31 = (F+1+F+F+1)+(F+F+1+F+F)+(1+F+F+1+F)+4(1+1) = 10F+13 ,$$

$$T22 = 4F^2+4(1+F+1+1+F+1)+2(1+F+1+1+F+1) = 4F^2+12F+24 ,$$

$$T21 = (F+F+F+1+1)+(F+F+F+F+1)+(1+F+F+F+F)+(1+1+F+F+F)+12 = 14F+18$$

$$T11 = 5F^2+(4+2)(F+1+1+1+F+1+1+F+1+F) = 5F^2+24F+36 . \qquad \text{(A-20)}$$

The desired average is, from (A-18) - (A-20),

$$324\ E\left(W_6^2\right) = 15F^2 + 144F + 202 . \qquad \text{(A-21)}$$

The variance of W_6 is then

$$V_6 = \frac{1}{324}\left(15F^2 + 144F - 23\right) . \qquad \text{(A-22)}$$

SPECIAL CASE K = 7

Continuing in the fashion established above, we now have

$$\phi_6^2 = 6600 \; , \quad \phi_5^2 = 5500+6611+2(6510) \; ,$$

$$\phi_4^2 = 4400+5511+6622+2(5410+6420+6521) \; ,$$

$$\phi_3^2 = 3300+4411+5522+6633+2(4310+5320+6330+5421+6431+6532) \; ,$$

$$\phi_2^2 = 2200+3311+4422+5533+6644+$$

$$+2(3210+4220+5320+6420+4321+5331+6431+5432+6442+6543) \; ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+6655+2(2110+3210+4310+5410+$$

$$+6510+3221+4321+5421+6521+4332+5432+6532+5443+6543+6554) \; . \qquad (A-23)$$

From (13),

$$W_7 = \frac{2}{49}\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_6^2\right) \qquad (A-24)$$

and therefore

$$\frac{2401}{4} W_7^2 = \phi_1^4 + \cdots + \phi_6^4 + 2\left(\phi_1^2 \; \phi_2^2 + \cdots \; \phi_5^2 \; \phi_6^2\right) \; . \qquad (A-25)$$

The required averages are as follows:

$$T66 = F^2 \; , \quad T65 = F+F = 2F \; , \quad T64 = F+1+F = 2F+1 \; ,$$

$$T63 = F+1+1+F = 2F+2 \; , \quad T62 = F+1+1+1+F = 2F+3 \; ,$$

$$T61 = F+1+1+1+1+F = 2F+4 \; , \quad T55 = F^2+F^2+4+2 = 2F^2+6 \; ,$$

$$T54 = (F+F+1)+(1+F+F) = 4F+2 \; , \; T53 = (F+1+F+1)+(1+F+1+F) = 4F+4 \; ,$$

$$T52 = (F+1+1+F+1)+(1+F+1+1+F) = 4F+6 \; ,$$

$$T51 = (F+1+1+1+F+F)+(F+F+1+1+1+F)+4 = 6F+10 \; ,$$

$$T44 = 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 \;,$$

$$T43 = (F+F+1+1)+(1+F+F+1)+(1+1+F+F) = 6F+6 \;,$$

$$T42 = (F+1+F+1+F)+(1+F+1+F+1)+(F+1+F+1+F)+4(1) = 8F+11 \;,$$

$$T41 = (3F+3)+(4F+2)+(3F+3)+4(1+1) = 10F+16 \;,$$

$$T33 = 4F^2+4(1+1+F+1+1+1)+2(1+1+F+1+1+1) = 4F^2+6F+30 \;,$$

$$T32 = (3F+2)+(3F+2)+(3F+2)+(3F+2)+4(1+1) = 12F+16 \;,$$

$$T31 = (3F+3)+(4F+2)+(4F+2)+(3F+3)+4(1+1+1) = 14F+22 \;,$$

$$T22 = 5F^2+(4+2)(1+F+1+1+1+F+1+1+F+1) = 5F^2+18F+42 \;,$$

$$T21 = 2(3F+3)+3(4F+2)+4(1+1+1+1) = 18F+28 \;,$$

$$T11 = 6F^2+(4+2)(F+1+1+1+1+F+1+1+1+F+1+1+F+1+F) = 6F^2+30F+60 \;. \tag{A-26}$$

The average of interest is, from (A-25) and (A-26),

$$\frac{2401}{4}\, E\left(W_7^2\right) = 21F^2 + 246F + 418 \;, \tag{A-27}$$

leading to variance

$$V_7 = \frac{4}{2401}\left(21F^2 + 246F - 23\right) \;. \tag{A-28}$$

SPECIAL CASE K = 8

This is the last case that we need to evaluate. We now have

$$\phi_7^2 = 7700 \ , \quad \phi_6^2 = 6600+7711+2(7610) \ ,$$

$$\phi_5^2 = 5500+6611+7722+2(6510+7520+7621) \ ,$$

$$\phi_4^2 = 4400+5511+6622+7733+2(5410+6420+7430+6521+7531+7632) \ ,$$

$$\phi_3^2 = 3300+4411+5522+6633+7744+$$

$$+2(4310+5320+6330+7430+5421+6431+7441+6532+7542+7643) \ ,$$

$$\phi_2^2 = 2200+3311+4422+5533+6644+7755+2(3210+4220+5320+6420+$$

$$+7520+4321+5331+6431+7531+5432+6442+7542+6543+7553+7654) \ ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+6655+7766+$$

$$+2(2110+3210+4310+5410+6510+7610+3221+4321+5421+6521+7621+$$

$$+4332+5432+6532+7632+5443+6543+7643+6554+7654+7665) \ . \qquad \text{(A-29)}$$

From (13) again,

$$W_8 = \frac{2}{64}\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_7^2\right) \ , \qquad \text{(A-30)}$$

giving

$$1024\ W_8^2 = \phi_1^4 + \cdots + \phi_7^4 + 2\left(\phi_1^2\ \phi_2^2 + \cdots + \phi_6^2\ \phi_7^2\right) \ . \qquad \text{(A-31)}$$

The averages needed are listed below.

$$T77 = F^2 \ , \quad T76 = 2F \ , \quad T75 = 2F+1 \ , \quad T74 = 2F+2 \ ,$$

$$T73 = 2F+3 \ , \quad T72 = 2F+4 \ , \quad T71 = 2F+5 \ ,$$

$$T66 = F^2+F^2+4+2 = 2F^2+6 \ , \quad T65 = (F+F+1)+(1+F+F) = 4F+2 \ ,$$

$$T64 = (F+1+F+1)+(1+F+1+F) = 4F+4 \ ,$$

$$T63 = (F+1+1+F+1)+(1+F+1+1+F) = 4F+6 \ ,$$

$$T62 = (F+1+1+1+F+1)+(1+F+1+1+1+F) = 4F+8 \ ,$$

$$T61 = (3F+4)+(3F+4)+4 = 6F+12 \ ,$$

$$T55 = 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 \ ,$$

$$T54 = (F+F+1+1)+(1+F+F+1)+(1+1+F+F) = 6F+6 \ ,$$

$$T53 = 3(2F+3) = 6F+9 \ ,$$

$$T52 = (3F+3)+(2F+4)+(3F+3)+4(1) = 8F+14 \ ,$$

$$T51 = (3F+4)+(4F+3)+(3F+4)+4(1+1) = 10F+19 \ ,$$

$$T44 = 4F^2+4(6)+2(6) = 4F^2+36 \ ,$$

$$T43 = (3F+2)+(2F+3)+(2F+3)+(3F+2)+4(1) = 10F+14 \ ,$$

$$T42 = 4(3F+3)+4(1+1) = 12F+20 \ ,$$

$$T41 = 2(3F+4)+2(4F+3)+4(1+1+1) = 14F+26 \ ,$$

$$T33 = 5F^2+4(2F+8)+2(2F+8) = 5F^2+12F+48 \ ,$$

$$T32 = 4(3F+3)+(4F+2)+4(1+1+1) = 16F+26 \ ,$$

$$T31 = 2(3F+4)+3(4F+3)+4(1+1+1+1) = 18F+33 \ ,$$

$$T22 = 6F^2+4(4F+11)+2(4F+11) = 6F^2+24F+66 \ ,$$

$$T21 = 2(3F+4)+4(4F+3)+4(1+1+1+1+1) = 22F+40 \ ,$$

$$T11 = 7F^2+4(6F+15)+2(6F+15) = 7F^2+36F+90 \ . \tag{A-32}$$

The desired average is therefore

$$1024 \ E\left(W_8^2\right) = 28F^2 + 384F + 772 \ , \tag{A-33}$$

giving variance

$$V_8 = \frac{1}{256}\left(7F^2 + 96F - 3\right) \ . \tag{A-34}$$

APPENDIX B. PROGRAM FOR ESTIMATION OF DISTRIBUTIONS OF W_K

```
10   T=1E6                    !  NUMBER OF TRIALS   "NUWC TR10237"
20   K=32                     !  NUMBER OF DATA POINTS, ARBITRARY
30   M=64                     !  FFT SIZE, M >= 2K-1, POWER OF 2
40   L=11000                  !  NUMBER OF LEVELS FOR DISTRIBUTION
50   Dw=.001                  !  INCREMENT IN W
60   Gr=1000                  !  GRID SPACING
70     PRINTER IS PRT
80   PRINT "K =";K;"   T =";T;"   Dw =";Dw;"   UNIFORM"
90     PRINTER IS CRT
100  DOUBLE T,K,M,L,M1,Ms,M2,K1,Ts,Ks  !  INTEGERS, NOT DP
110  DIM Cos(512),X(2048),Y(2048),V(30000)
120  M1=M-1
130  REDIM Cos(0:M/4),X(0:M1),Y(0:M1),V(0:L)
140  A=2.*PI/M
150  FOR Ms=0 TO M/4
160  Cos(Ms)=COS(A*Ms)     !  QUARTER-COSINE TABLE IN Cos(*)
170  NEXT Ms
180  M2=M/2
190  K1=K-1
200  T1=1./T
210  F=12./(K*M)              !  UNIT-VARIANCE UNIFORM
220  F=F*F                    !  RANDOM VARIABLES (x(subk))
230  Mu=K1/K                  !  EXACT MEAN OF WK
240  Mu1=Var=0.
250    Ta=TIMEDATE
260  FOR Ts=1 TO T
270  FOR Ks=0 TO K1
280  X(Ks)=RND-.5             !  ZERO MEAN
290  Y(Ks)=0.                 !  REAL INPUT
300  NEXT Ks
310  FOR Ks=K TO M1
320  X(Ks)=Y(Ks)=0.
330  NEXT Ks
340  CALL Fft14(M,Cos(*),X(*),Y(*))
350  S2=S4=0.
360  FOR Ms=1 TO M2-1         !  ZERO TO NYQUIST
370  X=X(Ms)
380  Y=Y(Ms)
390  A=X*X+Y*Y
400  S2=S2+A
410  S4=S4+A*A
420  NEXT Ms
430  X=X(0)
440  A=X(M2)
450  X=X*X
460  A=A*A
470  S2=X+A+2.*S2
480  S4=X*X+A*A+2.*S4
490  W=F*(M*S4-S2*S2)         !  WHITENESS MEASURE WK
```

```
500    Mu1=Mu1+W
510    Var=Var+(W-Mu)*(W-Mu)!  USE KNOWN MEAN Mu
520    Ms=INT(W/Dw)
530    Ms=MIN(Ms,L)
540    V(Ms)=V(Ms)+T1        !  INCREMENTAL PROBABILITIES
550    NEXT Ts
560      Tb=TIMEDATE
570      PRINTER IS PRT
580      PRINT (Tb-Ta)/3600;" HOURS"
590      PRINT
600    Mu1=Mu1/T             !  ESTIMATED MEAN OF WK
610    Var=Var/T             !  ESTIMATED VARIANCE OF WK
620    PRINT "Mu1 =";Mu1;"       Mu =";Mu
630    PRINT "Var =";Var
640    PRINT
650    PLOTTER IS "GRAPHICS"
660    GRAPHICS ON
670    WINDOW 0,L,-6,0
680    LINE TYPE 3
690    GRID Gr,1
700    LINE TYPE 1
710    C=0.
720    FOR Ms=0 TO L-1
730    C=C+V(Ms)             !  CDF OF WHITENESS MEASURE WK
740    IF C>0. THEN 760
750    GOTO 770
760    PLOT Ms+1,LGT(C)
770    NEXT Ms
780    PENUP
790    E=S1=S2=0.
800    FOR Ms=L TO 1 STEP -1
810    E=E+V(Ms)             !  EDF OF WHITENESS MEASURE WK
820    S1=S1+E
830    S2=S2+S1
840    IF E>0. THEN 860
850    GOTO 870
860    PLOT Ms,LGT(E)
870    NEXT Ms
880    PLOT 0,0
890    PENUP
900    Mu1=Dw*(.5+S1)        !  ESTIMATED MEAN OF WK
910    Mu2=2.*Dw*Dw*S2       !  SEE APPENDIX C
920    PRINT "Mu1 =";Mu1;"       Mu =";Mu
930    PRINT "Var =";Mu2-Mu*Mu ! ESTIMATED VARIANCE OF WK
940    PRINT
950      PRINTER IS CRT
960    PAUSE
970    END
980    !
990    SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
```

APPENDIX C. EVALUATION OF MOMENTS DIRECTLY FROM MEASURED EXCEEDANCE DISTRIBUTION

Let x be a positive random variable with probability density function p, cumulative distribution function (CDF) C, and exceedance distribution function (EDF) E. Let the measurements of these distributions be the interval probabilities

$$V_n = \text{Prob}(n\Delta \le x < (n+1)\Delta) \quad \text{for } 0 \le n . \tag{C-1}$$

Then

$$1 = \int_0^\infty dx\ p(x) = \sum_{n=0}^{\infty} \int_{n\Delta}^{(n+1)\Delta} dx\ p(x) = \sum_{n=0}^{\infty} V_n . \tag{C-2}$$

At the same time, we can express

$$V_n = C((n+1)\Delta) - C(n\Delta) = E(n\Delta) - E((n+1)\Delta) , \tag{C-3}$$

which can be inverted, leading respectively to EDF and CDF

$$E(n\Delta) = \text{Prob}(x \ge n\Delta) = \sum_{m=n}^{\infty} V_m \quad \text{for } n \ge 0 , \tag{C-4}$$

$$C(n\Delta) = \text{Prob}(x < n\Delta) = \sum_{m=0}^{n-1} V_m \quad \text{for } n \ge 1 . \tag{C-5}$$

There also follows

$$E(0) = 1 , \quad E((n+1)\Delta) = E(n\Delta) - V_n \quad \text{for } n \ge 0 , \tag{C-6}$$

or, as an alternative form to (C-4) if desired,

$$E(\Delta) = 1-V_0 , \quad E(2\Delta) = 1-V_0-V_1 , \quad E(3\Delta) = 1-V_0-V_1-V_2 , \ \ldots \tag{C-7}$$

The first two moments of random variable x can be developed as

$$\mu_1 = \int_0^\infty dx \; x \; p(x) = \int_0^\infty dx \; E(x) \cong \Delta\left[\frac{1}{2} E(0) + \sum_{n=1}^\infty E(n\Delta)\right] , \qquad (C-8)$$

and

$$\mu_2 = \int_0^\infty dx \; x^2 \; p(x) = 2 \int_0^\infty dx \; x \; E(x) \cong 2 \; \Delta^2 \sum_{n=1}^\infty n \; E(n\Delta) \; . \qquad (C-9)$$

These results can be rapidly evaluated by recursion. For $E((N+1)\Delta) = 0$, use

```
E=S1=S2=0.
FOR Ns=N TO 1 STEP -1
E=E+V(Ns)
S1=S1+E
S2=S2+S1
NEXT Ns
Mu1=Delta*(.5+S1)
Mu2=2.*Delta*Delta*S2
```
(C-10)

REFERENCES

[1] A. H. Nuttall, **On Generation of Random Numbers with Specified Distributions or Densities**, NUSC Technical Report 6843, Naval Underwater Systems Center, New London, CT, 1 December 1982

SUBJECT MATTER INDEX